On the excluded minors for quaternary matroids

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Abstract

This paper strengthens the excluded-minor characterization of GF(4)–representable matroids. In particular, it is shown that there are only finitely many 3–connected matroids that are not GF(4)–representable and that have no \(U_{2,6}−, U_{4,6}−, P_6−, F_7−\), or \((F_7)∗\)–minors. Explicitly, these matroids are all minors of \(S(5, 6, 12)\) with rank and corank at least 4, and \(P_8''\), the matroid that can be obtained from \(S(5, 6, 12)\) by deleting two elements, contracting two elements, and then relaxing the only pair of disjoint circuit–hyperplanes.

1 Introduction

Kahn and Seymour had conjectured that the excluded minors for the class of GF(4)–representable matroids are \(U_{2,6}, U_{4,6}, P_6,\) the non-Fano matroid \((F_7)\), and its dual; see [4, p 205]. It turns out that the complete set of excluded minors for GF(4)–representability contains two more matroids, namely \(P_8\) and \(P_8''\), see [1]. However, Kahn and Seymour were almost right, as we show in the following theorem.

**Theorem 1.1** If \(M\) is a 3–connected non-GF(4)–representable matroid, then either

(i) \(M\) has a \(U_{2,6}−, U_{4,6}−, P_6−, F_7−\), or \((F_7)∗\)–minor,

(ii) \(M\) is isomorphic to \(P_8''\), or

(iii) \(M\) is isomorphic to a minor of \(S(5, 6, 12)\) with rank and corank at least 4.

\(S(5, 6, 12)\), which is discussed in detail in [4], is the matroid that is represented over GF(3) by the following matrix.
\[
I_6 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 \\
1 & -1 & 1 & 0 & 1 \\
1 & -1 & -1 & 1 & 0
\end{pmatrix}
\]

Evidently \(S(5, 6, 12)\) is self–dual. Moreover, it has a 5–transitive automorphism group. \(P_8\) is the matroid that is obtained by deleting two elements and contracting two elements from \(S(5, 6, 12)\). Now \(P_8\) has a unique pair of disjoint circuit–hyperplanes and \(P_8''\) is obtained from \(P_8\) by relaxing both of these circuit–hyperplanes. These observations and those made before the theorem imply that the matroids satisfying (i), (ii), or (iii) are not quaternary.

The following corollary is a reformulation of Theorem 1.1. For a collection \(\mathcal{M}\) of matroids, we denote by \(EX(\mathcal{M})\) the class of matroids that have no minors isomorphic to a member of \(\mathcal{M}\).

**Corollary 1.2** \(EX(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)\) can be constructed by taking direct sums and 2–sums of copies of \(P_8''\), minors of \(S(5, 6, 12)\), and quaternary matroids.

We obtain Theorem 1.1 as a consequence of the excluded-minor characterization for quaternary matroids [1].

**Theorem 1.3** A matroid \(M\) is GF(4)–representable if and only if \(M\) has no minor isomorphic to any of \(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*, P_8, \) or \(P_8''\).

Theorem 1.1 is an immediate consequence of Theorem 1.3 and the following two theorems. Let \(\mathcal{M}\) be a minor–closed family of matroids. A matroid \(M\) in \(\mathcal{M}\) is called a splitter for \(\mathcal{M}\) if no 3–connected matroid in \(\mathcal{M}\) has a proper \(M\)–minor.

**Theorem 1.4** \(P_8''\) is a splitter for \(EX(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)\).

**Theorem 1.5** If \(M\) is a 3–connected matroid in \(EX(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)\), and \(M\) has a \(P_8\)–minor, then \(M\) is isomorphic to a minor of \(S(5, 6, 12)\).

Using Seymour’s Splitter Theorem [5,4], Theorems 1.4 and 1.5 can be proved by a finite case check. While we use this approach, we have endeavoured to find elegant techniques to reduce the number of cases.

**Theorem 1.6 (Splitter Theorem)** Let \(M\) and \(N\) be 3–connected matroids such that \(M\) is neither a wheel nor a whirl, \(N\) has at least four elements, and \(M\) contains a proper \(N\)–minor. Then \(M\) has an element \(x\) such that either \(M \setminus x\) or \(M/x\) is 3–connected with an \(N\)–minor.

We assume that readers are familiar with elementary notions in matroid theory, including representability, minors, duality, connectivity, and 1– and 2–sums. We use the notation and terminology of [4]. Figure 1 depicts some well–known matroids that are referred to in the paper. We will describe \(P_8\) and \(P_8''\) in more detail in the next section.
2 Dealing with $P_8$

In this section, we prove Theorem 1.5. We begin by describing some useful properties of $P_8$. This matroid has a very natural geometric representation; see Figure 2. This representation is obtained by rotating a face of the cube by 45 degrees. It is obvious from this description that $P_8$ has a transitive automorphism group. (However, there are automorphisms of $P_8$ that are not apparent from this description.) By this transitivity, all single-element contractions of $P_8$ are isomorphic to $P_8/8$, which is isomorphic to the matroid $P_7$ depicted in Figure 2. It is also not difficult to see that $P_8$ is self–dual; its dual is obtained by rotating the twisted face a further 90 degrees. Therefore every single-element deletion of $P_8$ is isomorphic to the dual of $P_7$.

$P_8''$ is the matroid obtained from $P_8$ by relaxing the circuit–hyperplanes $\{1,2,3,4\}$ and $\{5,6,7,8\}$. From this, it is readily seen that $P_8''$ is self–dual and has a transitive automorphism group, and that every single-element contraction is isomorphic to $P_7'$ (which is depicted in Figure 2).

![Figure 2: Some interesting matroids](image)

We use the following lemma [4, Proposition 11.2.16] and theorem [3] (see also [4, p. 367]). The matroid $J$ is a rank–4 self–dual matroid that is not isomorphic to $P_8$.

**Lemma 2.1** Let $M$ be a 3–connected matroid having rank and corank at least three. Then $M$ has a $U_{2,5}$–minor if and only if it has a $U_{3,5}$–minor.

**Theorem 2.2** If $M$ is a 3–connected matroid in $EX(U_{2,5},U_{3,5},M(K_4))$, then either $M$ is a whirl, $M$ is isomorphic to $J$, or $M$ is isomorphic to a minor of $S(5,6,12)$.
Corollary 2.3 If $M$ is a 3-connected matroid that is not isomorphic to a minor of $S(5,6,12)$, and $M$ has a $P_8$-minor, then there is a minor $N$ of either $M$ or $M^*$ and an element $x$ of $E(N)$ such that $N \setminus x$ is isomorphic to $P_8$, and $N$ contains either a $U_{2,5}$- or an $M(K_4)$-minor.

Proof Let $M'$ be a minimal 3-connected minor of $M$ that has a $P_8$-minor and a $U_{2,5}$-, $U_{3,5}$-, or $M(K_4)$-minor. By the Splitter Theorem, $M'$ has an element $x$ such that $M' \setminus x$ or $M'/x$ is 3-connected and has a $P_8$-minor. By duality, we may assume that $M' \setminus x$ is 3-connected and has a $P_8$-minor. By minimality, $M' \setminus x$ has no $U_{3,5}$-, $U_{2,5}$-, or $M(K_4)$-minor. Therefore, by Theorem 2.2, $M' \setminus x$ is isomorphic to a minor of $S(5,6,12)$.

As $M'$ has rank and corank at least 3, by Lemma 2.1, $M'$ has either a $U_{2,5}$- or $M(K_4)$-minor. Suppose that $M'$ has rank at least 5. Then there exists $y \in E(M') - x$ such that $M'/y$ has a $U_{2,5}$- or $M(K_4)$-minor. Now $M' \setminus x/y$ is isomorphic to a minor of $S(5,6,12)$ with rank and corank at least four. Hence, as $S(5,6,12)$ has a 3-transitive automorphism group, $M' \setminus x/y$ is 3-connected and has a $P_8$-minor. Now $M'/y$ is an extension of the 3-connected matroid $M'/y \setminus x$, and, since $M'/y \setminus x$ has no $U_{3,5}$- or $M(K_4)$-minor, $x$ is not in parallel with any element of $M'/y$. Hence $M'/y$ is 3-connected. Moreover, $M'/y$ has a $P_8$-minor and a $U_{2,5}$- or $M(K_4)$-minor, contradicting the minimality of $M'$. Therefore, $M'$ has rank 4.

A similar argument to that in the last paragraph establishes that the corank of $M'$ is 5. Therefore, taking $N$ to be equal to $M'$, we see that the theorem holds.

Theorem 1.5 is implied by Corollary 2.3 and the following two results.

Lemma 2.4 If $M \setminus x = P_8$ and $M$ has an $M(K_4)$-minor, then $M$ has a $U_{2,5}$-, $F_7^-$, or $(F_7^-)^*$-minor.

Lemma 2.5 If $M \setminus x = P_8$ and $M$ has a $U_{2,5}$-minor, then $M$ has a $U_{2,6}$-, $U_{4,6}$-, $P_6$, $F_7^-$, or $(F_7^-)^*$-minor.

Proof of Lemma 2.4: Suppose that $M$ is in $EX(U_{2,5}, F_7^-, (F_7^-)^*)$. Since $M$ has no $U_{2,5}$-minor and $M$ has rank and corank at least 3, Lemma 2.1 implies that $M$ has no $U_{3,5}$-minor.

We show next that $E(M) - x$ contains an element $y$ such that $M/y$ has an $M(K_4)$-minor. Suppose not. Then, as $M$ has corank 4, $M/x$ has an $M(K_4)$-minor. Therefore, $E(M) - x$ contains elements $a$ and $b$ such that $M/x \setminus a, b \cong M(K_4)$. Now $M \setminus a, b, x$ is isomorphic to a matroid obtained from $P_7^*$ by deleting an element. Thus $M \setminus a, b, x$ has either two or three disjoint series pairs. Now $M \setminus a, b$ has no series pairs, otherwise we could contract an element other than $x$ leaving an $M(K_4)$-minor. Therefore, $x$ is in either two or three 3-element cocircuits of $M \setminus a, b$, and any two such cocircuits have only $x$ in common. Thus, as $M \setminus a, b/x \cong M(K_4)$, it follows that $M \setminus a, b$ is isomorphic to $F_7^*$ or $(F_7^-)^*$. Now $M$ certainly has no $(F_7^-)^*$-minor. Thus $M \setminus a, b \cong F_7^*$ and so, for every $y$ in $E(M) \setminus \{x, a, b\}$, the matroid $M/y$ has an $M(K_4)$-minor. This contradiction implies that there is, indeed, an element $y$ of $E(M) - x$ such that $M/y$ has an $M(K_4)$-minor. As $P_8$ has a transitive automorphism group, we may assume that $y = 8$.
Now $M/8$ is an extension of $P_7$ that has no $U_{3,5}$-minor. Furthermore, as $P_7$ has no $M(K_4)$-minor and $M/8$ has an $M(K_4)$-minor, $M/8$ is 3-connected. It is not difficult to check that there are just three 3-connected extensions of $P_7$ that have no $U_{3,5}$- and hence no $U_{2,5}$-minor; these are depicted in Figure 3. Note that $M_2 \setminus a \cong F_7^-$, and $M_3$ has no $M(K_4)$-minor. Hence $M/8 \cong M_1$. Thus, in $M$, the element $x$ lies on the intersection of the planes spanned by the circuit–hyperplanes $\{2, 4, 5, 8\}$ and $\{1, 3, 7, 8\}$ of $P_8$. We may assume that $x$ is not in the plane of $M$ spanned by $\{1, 2, 3, 4\}$, otherwise $M \setminus 1, 2 \cong (F_7^-)^\times$. To see this, observe that if $x$ is in the plane spanned by $\{1, 2, 3, 4\}$, then the planes $\{1, 2, 3, 4, x\}$ and $\{2, 4, 5, 8, x\}$ of $M$ imply that $\{2, 4, x\}$ is a circuit of $M$. Similarly, the planes $\{1, 2, 3, 4, x\}$ and $\{1, 3, 7, 8, x\}$ of $M$ imply that $\{1, 3, x\}$ is a circuit of $M$. We deduce that $\{2, 4, 6, 7, x\}, \{1, 3, 5, 6, x\}, \{3, 4, 5, 7\}$, and $\{5, 6, 7, 8\}$ are hyperplanes of $M$, and it is now not difficult to obtain the contradiction that $M \setminus 1, 2 \cong (F_7^-)^\times$.

![Figure 3: Extensions of $P_7$ with no $U_{3,5}$-minor](image)

Next we show that either $M/1$ or $M/3$ is 3-connected. Assume the contrary and note that $M/1 \setminus x$ is isomorphic to $P_7$, which is 3-connected. Now, as $x$ lies on the plane spanned by $\{1, 3, 7, 8\}$, since $M/1$ is not 3-connected, $x$ is parallel to 3, 7, or 8 in $M/1$. However, as $x$ is not in the plane spanned by $\{1, 2, 3, 4\}$, the element $x$ is not parallel to 3 in $M/1$. Also $x$ is not parallel to 8 in $M/1$ since $M/8$ is 3-connected. Thus $x$ is parallel to 7 in $M/1$, and hence $\{1, x, 7\}$ is a line in $M$. By symmetry, as $M/3$ is not 3-connected, $\{3, x, 7\}$ is a line in $M$. Thus $\{1, 3, 7\}$ is a line in $P_8$. This contradiction completes the proof that either $M/1$ or $M/3$ is 3-connected. But $P_8$ has an automorphism that swaps 1 and 2 with 3 and 4, respectively, while fixing all other elements. Therefore, we may assume that $M/1$ is 3-connected.

Now $M/1$ is a 3-connected extension of $P_7$ with no $U_{3,5}$-minor. Furthermore, the point $x$ of the extension is on a 4-point line with the tip of $P_7$. Thus, $M/1$ is isomorphic to $M_2$ of Figure 3. However, $M_2 \setminus a \cong F_7^-$; a contradiction. ■

To prove Lemma 2.5, we employ methods used in [1]. For a field $\mathbb{F}$, two $r \times n$ matrices over $\mathbb{F}$ whose sequences of column labels coincide are equivalent $\mathbb{F}$-representations of a matroid if one matrix can be obtained from the other by elementary row operations, column scalings, and applying automorphisms of $\mathbb{F}$. A matroid is uniquely representable over $\mathbb{F}$ if any two $\mathbb{F}$-representations of it are equivalent. If the $r \times n$ matrix $(I_r|D)$ with columns labelled $e_1, e_2, \ldots, e_n$ represents a matroid $M$ over $\mathbb{F}$, it is common to abbreviate this $\mathbb{F}$-representation by specifying just the matrix $D$ labelling its rows and columns by $e_1, e_2, \ldots, e_r$ and $e_{r+1}, e_{r+2}, \ldots, e_n$, respectively. Such a
matrix $D$ will be called a standard $F$–representation of $M$.

A matroid is \textit{stable} if it cannot be expressed as the direct sum or 2–sum of two nonbinary matroids. For our purposes, the most important examples of stable matroids are those matroids which simplify to 3–connected matroids. Kahn [2] proved that a quaternary matroid has a unique GF(4)–representation if and only if it is stable. The following corollary of Kahn’s theorem is established in [1].

\textbf{Proposition 2.6} Let $M$ be a matroid, and $u, v$ be a coindependent pair of elements of $M$ such that $M/u$, $M/v$, and $M/u,v$ are all stable, and $M/u,v$ is connected and nonbinary. If $M/u$ and $M/v$ are both quaternary, then there is a unique quaternary matroid $N$ such that $N/u = M/u$ and $N/v = M/v$.

\textbf{Proof of Lemma 2.5:} Since $M$ has rank four, $E(M) - x$ contains an element $a$ such that $M/a$ has a $U_{2,5}$–minor. Now $M/a$ is isomorphic to an extension of $P_7$. We assert that $E(M) - x - a$ contains an element $b$ such that $M/a,b$ has a $U_{2,5}$–restriction. Suppose not. Then $M/a, x$ has a $U_{2,5}$–restriction. Hence there are at least three elements of $M/a$ that are not on a 3– or 4–point line with $x$. Let $b$ be one of these points, other than the tip of $M/a \setminus x$. Then $M/a, b$ has a $U_{2,5}$–restriction, as asserted.

We may assume that $M$ has no $U_{2,6}$–minor, so $M/a,b$ simplifies to $U_{2,5}$. Hence $M/a,b$ is stable. Now $M/a$ and $M/b$ are both isomorphic to extensions of $P_7$. Hence $M/a$ and $M/b$ are both stable. We may assume that $M/a$ and $M/b$ are both quaternary. Then, by Proposition 2.6, there is a unique quaternary matroid $N$ such that $N/a = M/a$ and $N/b = M/b$.

Every pair of points of $P_8$ is equivalent, under automorphism, to either $(1, 2), (1, 3)$, or $(1, 8)$. Now $P_8/1, 3$ is binary, so no extension of $P_8/1, 3$ has a $U_{2,5}$–restriction. Hence we may assume that $(a, b)$ is either $(1, 2)$ or $(1, 8)$.

\textbf{Case 1:} Suppose that $a = 1$ and $b = 2$. Note that $M \setminus x/1, M \setminus x/2$, and $M \setminus x/1, 2$ are all stable, and that $M \setminus x/1, 2$ is connected and nonbinary. So, by Proposition 2.6, $N \setminus x$ is the unique quaternary matroid such that $N \setminus x/1 = M \setminus x/1$ and $N \setminus x/2 = M \setminus x/2$. Therefore, $N \setminus x$ has the following standard GF(4)–representation (where $w^2 = w + 1$).

\[
\begin{pmatrix}
3 & 4 & 5 & 7 \\
1 & 0 & 1 & 1 & w \\
2 & 1 & 0 & w & 1 \\
6 & 1 & 1 & 1 & 1 \\
8 & w + 1 & w + 1 & 1 & 1
\end{pmatrix}
\]

Recall that $N/1, 2$ has a $U_{2,6}$–minor, so a GF(4)–representation for $N$ can be obtained by appending the column $(\alpha, \beta, 1, w)^T$ to the above matrix, where $\alpha$ and $\beta$ are yet to be determined.

First suppose that $\alpha = \beta = 0$, and consider the following standard GF(4)–representation of $N \setminus 6$.

\[
\begin{pmatrix}
3 & 4 & 5 & 7 \\
1 & 0 & 1 & 1 & w \\
2 & 1 & 0 & w & 1 \\
7 & 1 & 1 & 1 & 1 \\
8 & 1 & 1 & w + 1 & w + 1
\end{pmatrix}
\]
We see that \(N \setminus 6/1, N \setminus 6/2\), and \(N \setminus 6/1, 2\) are all stable, connected, and nonbinary. Furthermore, \(N \setminus 6/1 = M \setminus 6/1\), \(N \setminus 6/2 = M \setminus 6/2\), and \(N \setminus 6/1, 2 = M \setminus 6/1, 2\). So, by Proposition 2.6, \(N \setminus 6\) is the unique GF(4)-representable matroid such that \(N \setminus 6/1 = M \setminus 6/1\) and \(N \setminus 6/2 = M \setminus 6/2\). Now, \(\{x, 6, 8\}\) is a triangle of \(N\). It is also a triangle of \(M\) otherwise both \(\{1, x, 6, 8\}\) and \(\{2, x, 6, 8\}\) are circuits of \(M\) implying the contradiction that \(\{1, 2, 6, 8\}\) is dependent in \(M\). Moreover, \(\{5, 6, 7, 8\}\) is a circuit in \(M\) but not in \(N\). Hence, \(\{5, 7, x, 8\}\) is dependent in \(M \setminus 6\) although it is independent in \(N \setminus 6\). In particular, \(N \setminus 6 \neq M \setminus 6\), so, by uniqueness, \(M \setminus 6\) is not GF(4)-representable. Now \(M \setminus 4, 6, 7, 1 = N \setminus 4, 6, 7, 1 \cong U_{3, 5}\), so \(M \setminus 6\) is not isomorphic to \(P_8\) since the last matroid is ternary. Also \(M \setminus 6, x \cong P_7^*, \) so \(M \setminus 6\) is not isomorphic to \(P_8^\prime\). Therefore, by Theorem 1.3, \(M \setminus 6\) has a \(U_{2, 6-}, U_{4, 6-}, P_6-, F_7^-, (F_7^*)^-\)-minor, as required.

We may now assume that either \(\alpha \neq 0\) or \(\beta \neq 0\). Using the automorphism of \(P_8\) that swaps 1, 4, and 5 with 2, 3, and 7, respectively, we may assume that \(\alpha \neq 0\). Then, it is easy to check that \(N \setminus 4/1, N \setminus 4/2\) and \(N \setminus 4/1, 2\) are all stable, connected, and nonbinary. Furthermore, \(N \setminus 4/1 = M \setminus 4/1\), \(N \setminus 4/2 = M \setminus 4/2\), and \(N \setminus 4/1, 2 = M \setminus 4/1, 2\). So, by Proposition 2.6, \(N \setminus 4\) is the unique GF(4)-representable matroid such that \(N \setminus 4/1 = M \setminus 4/1\) and \(N \setminus 4/2 = M \setminus 4/2\). Now, \(\{5, 6, 7, 8\}\) is a circuit in \(M \setminus 4\) but not in \(N \setminus 4\). In particular, \(N \setminus 4 \neq M \setminus 4\), so, by uniqueness, \(M \setminus 4\) is not GF(4)-representable. However, \(M \setminus 4, x \cong P_7^*, \) so \(M \setminus 4\) is not isomorphic to \(P_8^\prime\). Also \(M \setminus 4, 5/1, 2 = N \setminus 4, 5/1, 2 \cong U_{2, 5}\), so \(M \setminus 4\) is not isomorphic to \(P_8\). Therefore, by Theorem 1.3, \(M \setminus 4\) has a \(U_{2, 6-}, U_{4, 6-}, P_6-, F_7^-, (F_7^*)^-\)-minor, as required.

**Case 2:** Suppose that \(a = 1\) and \(b = 8\). Note that \(M \setminus x/1, M \setminus x/8, \) and \(M \setminus x/1, 8\) are all stable, and that \(M \setminus x/1, 8\) is connected and nonbinary. So, by Proposition 2.6, \(N \setminus x\) is the unique quaternary matroid such that \(N \setminus x/1 = M \setminus x/1\) and \(N \setminus x/8 = M \setminus x/8\). Therefore, \(N \setminus x\) has the following standard GF(4)-representation.

\[
\begin{array}{cccc}
3 & 4 & 5 & 7 \\
1 & 0 & 1 & 1 \\
2 & 1 & 0 & w + 1 \\
6 & 1 & 1 & 1 \\
8 & w & w & 1 \\
\end{array}
\]

Recall that \(N/1, 8\) has a \(U_{2, 5}\)-minor, so a GF(4)-representation for \(N\) can be obtained by appending the column \((\alpha, w, 1, \beta)^T\) to the above matrix, where \(\alpha\) and \(\beta\) are yet to be determined.

Now, \(N \setminus 5, x/1\) and \(N \setminus 5, x/8\) are both 3-connected. So, it is easy to check that \(N \setminus 5/1, N \setminus 5/8\) and \(N \setminus 5/1, 8\) are all stable, connected, and nonbinary. Furthermore, \(N \setminus 5/1 = M \setminus 5/1, \) \(N \setminus 5/8 = M \setminus 5/8, \) and \(N \setminus 5/1, 8 = M \setminus 5/1, 8\). So, by Proposition 2.6, \(N \setminus 5\) is the unique GF(4)-representable matroid such that \(N \setminus 5/1 = M \setminus 5/1\) and \(N \setminus 5/8 = M \setminus 5/8\). Now, \(\{2, 4, 6, 7\}\) is a circuit in \(M \setminus 5\) but not in \(N \setminus 5\). In particular, \(N \setminus 5 \neq M \setminus 5\), so, by uniqueness, \(M \setminus 5\) is not GF(4)-representable. However, \(M \setminus 5, x \cong P_7^*, \) so \(M \setminus 5\) is not isomorphic to \(P_8^\prime\). It is left to the reader to check that, for any \(\alpha \in GF(4), \) the matroid \(N/8 \setminus 5\) is not isomorphic to \(P_7\). Hence, \(M/8 \setminus 5\) is not isomorphic to \(P_7\), so \(M \setminus 5\) is not isomorphic to \(P_8\). Therefore, by Theorem 1.3, \(M \setminus 5\) has a \(U_{2, 6-}, U_{4, 6-}, P_6-, F_7^-, (F_7^*)^-\)-minor, as required.
3 Dealing with $P''_8$

In this section, we prove Theorem 1.4. The techniques are very similar to those used in the previous section.

Since $P''_8$ is self-dual, it suffices to prove that every 3-connected single-element extension of $P''_8$ contains a $U_{2,6}$, $U_{4,6}$, $P_6^-$, $F_7^-$, or $(F_7^-)^*$-minor. Suppose not. Then there is a 3-connected matroid $M$ in $EX(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)$ such that $M \setminus x = P''_8$.

3.1 $M/1, 3$ has no $U_{2,5}$-restriction.

Suppose to the contrary that $M/1, 3$ has a $U_{2,5}$-restriction. It is readily seen that $M/1$, $M/3$, and $M/1, 3$ are all stable, connected, and nonbinary. Then, by Proposition 2.6, there is a unique quaternary matroid $N$ such that $N/1 = M/1$ and $N/3 = M/3$. It is easily checked that $N \setminus x$ is uniquely representable over $GF(4)$, and has the following matrix as a standard $GF(4)$-representation.

\[
\begin{pmatrix}
2 & 4 & 6 & 7 \\
1 & 1 & 1 & 1 \\
3 & 1 & w+1 & 1 \\
5 & 1 & w & 1 & 0 \\
8 & 1 & w+1 & 0 & 1 \\
\end{pmatrix}
\]

Recall that $N/1, 3$ has a $U_{2,5}$-minor, so a $GF(4)$-representation for $N$ can be obtained by appending the column $(\alpha, \beta, 1, w+1)^T$ to the above matrix, where $\alpha$ and $\beta$ are yet to be determined.

It is easy to check that $N \setminus 4/1$, $N \setminus 4/3$, and $N \setminus 4/1, 3$ are all stable, connected, and nonbinary. So, by Proposition 2.6, $N \setminus 4$ is the unique $GF(4)$-representable matroid such that $N \setminus 4/1 = M \setminus 4/1$ and $N \setminus 4/3 = M \setminus 4/3$. Now, $\{5, 6, 7, 8\}$ is a circuit in $N \setminus 4$ but not in $M \setminus 4$. In particular, $N \setminus 4 \neq M \setminus 4$, so, by uniqueness, $M \setminus 4$ is not $GF(4)$-representable. However, $M \setminus 4, x \cong (P''_7)^*$, so $M \setminus 4$ is not isomorphic to $P_8$. Therefore, by Theorem 1.3, either $M \setminus 4$ is isomorphic to $P''_8$, or $M \setminus 4$ has a $U_{2,6}$, $U_{4,6}$, $P_6^-$, $F_7^-$, or $(F_7^-)^*$-minor. Thus, we may assume that $M \setminus 4$ is isomorphic to $P''_8$. In particular, $M \setminus 2, 4$ is isomorphic to $(P''_7)^*$. Therefore, neither $M \setminus 2, 4/3$ nor $M \setminus 2, 4/1$ is isomorphic to $M(K_4)$. Consequently, $\alpha \in \{w, w+1\}$ and $\beta \in \{1, w\}$. Therefore, $(M \setminus 2, 4/3)^*$ and $(M \setminus 2, 4/1)^*$ are both isomorphic to $Q_6$ where, in each case, the two 3-point lines are $\{x, 5, 6\}$ and $\{x, 7, 8\}$. Hence, $(M \setminus 2, 4)^*$ has at most three 3-point lines, and so it is not isomorphic to $P''_7$. This contradiction proves (3.1).

3.2 $M/1, 8$ has no $U_{2,5}$-restriction.

Suppose to the contrary that $M/1, 8$ has a $U_{2,5}$-restriction. It is readily seen that $M/1$, $M/8$, and $M/1, 8$ are all stable, connected, and nonbinary. Then, by Proposition 2.6, there is a unique quaternary matroid $N$ such that $N/1 = M/1$ and $N/8 = M/8$. It is easily checked that $N \setminus x$ is
uniquely representable over $GF(4)$, and has following matrix as a standard $GF(4)$–representation.

\[
\begin{pmatrix}
2 & 4 & 6 & 7 \\
1 & 1 & 1 & w \\
3 & 1 & w & 1 \\
5 & 1 & w + 1 & 1 \\
8 & 1 & w & 0 & 1
\end{pmatrix}
\]

Recall that $N/1, 8$ has a $U_{2,5}$-minor, so a $GF(4)$–representation for $N$ can be obtained by appending the column $(\alpha, 1, w, \beta)^T$ to the above matrix, where $\alpha$ and $\beta$ are yet to be determined.

Now, $N \setminus 2, x/1$ and $N \setminus 2, x/8$ are both 3–connected. So, it is easy to check that $N \setminus 2/1$, $N \setminus 2/8$ and $N \setminus 2/1, 8$ are all stable, connected, and nonbinary. So, by Proposition 2.6, $N \setminus 2$ is the unique $GF(4)$–representable matroid such that $N \setminus 2/1 = M \setminus 2/1$ and $N \setminus 2/8 = M \setminus 2/8$. Now, $\{3, 4, 5, 7\}$ is a circuit in $M \setminus 2$ but not in $N \setminus 2$. In particular, $N \setminus 2 \neq M \setminus 2$, so, by uniqueness, $M \setminus 2$ is not $GF(4)$–representable. However, $M \setminus 2, x \cong (P_7')^*$, so $M \setminus 2$ is not isomorphic to $P_8$. Therefore, by Theorem 1.3, either $M \setminus 2$ is isomorphic to $P_8'$, or $M \setminus 2$ has a $U_{2,6}$, $U_{4,6}$, $P_6^-$, $F_7^-$, or $(F_7^-)^*$–minor. Thus, we may assume that $M \setminus 2$ is isomorphic to $P_8'$. In particular, $M \setminus 2/1$ is isomorphic to $P_7'$. It is left to the reader to check that this implies that $\beta = w + 1$. Then $M/1, 3$ has a $U_{2,3}$–restriction, contradicting (3.1). This proves (3.2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{An extension of $P_7'$}
\end{figure}

Let $R$ be the matroid depicted in Figure 4.

\subsection{3.3 If $M/1$ is 3–connected, then it is isomorphic to $R$ where the 4–point line is either $\{3, 7, 8, x\}$ or $\{3, 5, 6, x\}$.}

Suppose $M/1$ is 3–connected. Then it is isomorphic to a 3–connected extension of $P_7'$. Now 2 and 4 are each on only one 3–point line of $M/1 \setminus x$. So, since neither $M/1, 2 \setminus x$ nor $M/1, 4 \setminus x$ has a $U_{2,6}$–restriction, each of 2 and 4 is on some 3– or 4–point line of $M/1$ with $x$. By (3.1), 3 is also on some 3– or 4–point line of $M/1$ with $x$. By (3.2) and symmetry, each of $5, 6, 7, 8$ is on some 3– or 4–point line of $M/1$ with $x$. Hence $x$ is on some 3– or 4–point line with every other element of $M/1$. It follows that $x$ is on one 4–point line and two 3–point lines in $M/1$. There is, up to isomorphism, just one such extension of $P_7'$, namely $R$. This proves (3.3).
If \( x \) lies on three 3-point lines in \( M \), say \( \{x, a_1, a_2\}, \{x, b_1, b_2\} \) and \( \{x, c_1, c_2\} \), then \( \{a_1, a_2, b_1, b_2\}, \{a_1, a_2, c_1, c_2\} \), and \( \{b_1, b_2, c_1, c_2\} \) are all hyperplanes of \( P''_8 \). However, \( P''_8 \) has no three such hyperplanes, so \( x \) is on at most two 3-point lines of \( M \). Furthermore, if \( \{x, a_1, a_2\} \) and \( \{x, b_1, b_2\} \) are 3-point lines, then \( \{a_1, a_2, b_1, b_2\} \) is a hyperplane of \( P''_8 \). Therefore, since any two points of \( P''_8 \) lie on some circuit–hyperplane of \( P''_8 \), if \( M \) has any 3-point lines, then \( M \) is obtained by adding \( x \) to some circuit–hyperplane of \( P''_8 \), and all 3-point lines are contained in that hyperplane. The automorphisms of \( P''_8 \) act transitively on its circuit–hyperplanes, so we may assume that the 3-point lines of \( M \) use only points from the set \( \{x, 2, 4, 6, 7\} \). Therefore, \( M/1, M/3, M/5, \) and \( M/8 \) are all 3-connected. Therefore, by (3.3) and symmetry, each of these matroids is isomorphic to \( R \). Since \( M/1 \) is isomorphic to \( R \), either \( \{x, 1, 3, 5, 6\} \) or \( \{x, 1, 3, 7, 8\} \) is a hyperplane of \( M \). Using the automorphism of \( P''_8 \) that swaps 4, 5, and 6 with 2, 8, and 7, respectively, we may assume that \( \{x, 1, 3, 5, 6\} \) is a hyperplane of \( M \). Then \( M/5 \) is not isomorphic to \( R \). This contradiction completes the proof of Theorem 1.4.

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References


