Unitary Representations of Lie Groups with Reflection Symmetry

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Abstract

We consider the following class of unitary representations $\pi$ of some (real) Lie group $G$ which has a matched pair of symmetries described as follows: (i) Suppose $G$ has a period-2 automorphism $\tau$, and that the Hilbert space $H(\pi)$ carries a unitary operator $J$ such that $J\pi = (\pi \circ \tau)J$ (i.e., self-similarity). (ii) An added symmetry is implied if $H(\pi)$ further contains a closed subspace $K_0$ having a certain order-covariance property, and satisfying the $K_0$-restricted positivity: $\langle v | Jv \rangle \geq 0$, $\forall v \in K_0$, where $\langle \cdot | \cdot \rangle$ is the inner product in $H(\pi)$. From (i)-(ii), we get an induced dual representation of an associated dual group $G^\circ$. All these properties, self-similarity, order-covariance, and positivity, are satisfied in a natural context when $G$ is semisimple and hermiteian; but when $G$ is the $(ax + b)$-group, or the Heisenberg group, positivity is incompatible with the other two axioms for the infinite-dimensional irreducible representations. We describe a class of $G$, containing the latter two, which admits a classification of the possible spaces $K_0 \subset H(\pi)$ satisfying the axioms of self-similarity and order-covariance.

1 Introduction

We consider a class of unitary representations of a Lie group $G$ which possess a certain reflection symmetry defined as follows: If $\pi$ is a representation of $G$ in some Hilbert space $H$, we introduce the following three structures:

i) $\tau \in \text{Aut}(G)$ of period 2;

ii) $J : H \to H$ is a unitary operator of period 2 such that $J\pi(g)J^* = \pi(\tau(g))$, $g \in G$ (this will hold if $\pi$ is of the form $\pi_+ \oplus \pi_-$ with $\pi_+$ and $\pi_- \circ \tau$ unitarily equivalent); it will further be assumed that there is a closed subspace $K_0 \subset H$ which is invariant under $\pi(H)$, $H = G^\circ$, or more generally, under an open subgroup of $G^\circ$;

iii) positivity is assumed in the sense that $\langle v | J(v) \rangle \geq 0$, $v \in K_0$.

Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\mathfrak{h}$ be the Lie algebra of the fixed-point subgroup $G^\circ = \{ g \in G \mid \tau(g) = g \}$. Let $\mathfrak{q} = \{ Y \in \mathfrak{g} \mid \tau(Y) = -Y \}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}. $$

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Let $H$ be a closed subgroup of $G$, $G_o^c < H < G^c$. Assume there is an $H$-invariant, closed, and generating convex cone $C$ in $\mathfrak{q}$ (i.e., $C - C = \mathfrak{q}$) such that $C^o$ consists of hyperbolic elements. We assume that $S(C) = H \exp C$ is a closed semigroup in $G$ which is homeomorphic to $H \times C$, and that
\[
H \times C^o \ni (h, Y) \mapsto h \exp Y \in S^o
\]
is a diffeomorphism.

We shall consider closed subspaces $K_0 \subset H(\pi)$, where $H(\pi)$ is the Hilbert space of $\pi$, such that $K_0$ is invariant under $\pi(S^o)$. Let $J : H(\pi) \to H(\pi)$ be a unitary intertwining operator for $\pi$ and $\pi \circ \tau$ such that $J^2 = \text{id}$. We assume that $K_0$ may be chosen such that $\|v\|^2 := \langle v | Jv \rangle \geq 0$ for all $v \in K_0$. We will always assume our inner product conjugate linear in the first argument. We form, in the usual way, the Hilbert space $K = \left( K_0 / N \right)$ by dividing out with $N = \{ v \in K_0 | \langle v | Jv \rangle = 0 \}$ and completing in the norm $\| \cdot \|_J$. (This is of course a variation of the Gelfand-Naimark-Segal (GNS) construction.) With the properties of $(G, \pi, H(\pi), K)$ as stated, we show, using the Lüscher-Mack theorem, that the simply connected Lie group $G^c$ with Lie algebra $\mathfrak{g}^c = \mathfrak{h} \oplus i \mathfrak{q}$ carries a unitary representation $\pi^c$ on $K$ such that $\{ \pi^c(h \exp(iY)) | h \in H, Y \in C^o \}$ is obtained from $\pi$ by passing the corresponding operators $\pi(h \exp Y)$ to the quotient $K_0 / N$. In fact, when $Y \in C$, the selfadjoint operator $d\pi(Y)$ on $K$ has spectrum contained in $(-\infty, 0]$.

As in Corollary 3.4, we show that in the case where $C$ extends to an $G^c$ invariant regular cone in $i\mathfrak{g}^c = i\mathfrak{h} \oplus i \mathfrak{q}$ and $\pi^c$ is injective, then each $\pi^c$ (as a unitary representation of $G^c$) must be a direct integral of highest-weight representations of $G^c$. The examples show that one can relax the condition in different ways, i.e., one can avoid using the Lüscher-Mack theorem by instead constructing local representations and using only cones that are neither generating nor $H$-invariant.

Let us outline the plan by a simple example. Let $G = \text{SL}(2, \mathbb{R})$, and let $P$ be the parabolic subgroup
\[
P = \left\{ p(a, x) = \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} | a \in \mathbb{R}^+, x \in \mathbb{R} \right\}.
\]
For $s \in \mathbb{C}$, let $\pi_s$ be the representation of $G$ acting by $[\pi_s(a)f](b) = f(a^{-1}b)$ on the space $H_s$ of functions $f : G \to \mathbb{C}$,
\[
f(g p(a, x)) = |a|^{-s-1} f(g), \quad \int_{SO(2)} |f(k)|^2 dk < \infty,
\]
and with inner product
\[
\langle f | g \rangle = \int_{SO(2)} \overline{f(k)} g(k) \, dk,
\]
i.e., $\pi_s$ is the principal series representation of $G$ with parameter $s$. The representations $\pi_s$ are unitary in the above Hilbert-space structure as long as $s \in i\mathbb{R}$. For defining a unitary structure for other parameters we need the intertwining operator $A_s : H_s \to H_{-s}$ defined by
\[
A_s(f)(g) = \int_{-\infty}^{\infty} f(g w \tilde{n}_y) \, dy
\]
for $\Re s \geq 0$ and then generally by analytic continuation. Here $w$ is the Weyl group element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tilde{n}_y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$.

By restriction to $\tilde{N} = \{ \tilde{n}_y | y \in \mathbb{R} \}$ we can also realize the representations $\pi_s$ on $\mathbb{R} \cong \tilde{N}$, $y \mapsto \tilde{n}_y$. Using that
\[
\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha x \\ \alpha y & \alpha y x + \alpha^{-1} \end{pmatrix}
\]
we have \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{NP} \) if and only if \( a \neq 0 \), and in that case
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pi_{c/|a| P}(a, b/|a|).
\]
(1.1)

Thus the intertwinor \( A_s \) becomes the singular integral operator
\[
A_s f(x) = \int_{-\infty}^{\infty} f(y) |x - y|^{s-1} dy.
\]

In the new inner product
\[
\langle f | A_s g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y)|x - y|^{s-1} dx dy
\]
the representation \( \pi_s \), which is given by
\[
\left[ \pi_s \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) f \right](x) = | -bx + d|^{-s+1} f \left( \frac{ax - c}{-bx + d} \right),
\]
is unitary for \( 0 < s < 1 \). Notice that we now denote by \( \mathbf{H}_s \) the new Hilbert space with the inner product \( < \cdot | A_s(\cdot) > \).

Define an involution \( \tau \) on \( G \) by
\[
\tau \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) := \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.
\]
(1.2)

The group \( H \) is given by
\[
H = \pm \left\{ h_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \Big| t \in \mathbb{R} \right\}
\]
and the space \( q \) is
\[
q = \left\{ q(r, s) := \begin{pmatrix} r & s \\ -s & -r \end{pmatrix} \Big| r, s \in \mathbb{R} \right\}.
\]

Take
\[
C := \{ q(r, s) \mid r \pm s \geq 0, r \geq 0 \} = \operatorname{conv}(\mathbb{R}_+ \operatorname{Ad}(H)q(1, 0))
\]
as a generating cone. The Cartan involution \( \theta \) is given by \( a \mapsto a^{-1} = waw^{-1} \) and the corresponding maximal compact subgroup is \( \text{SO}(2) \). Define
\[
Jf(a) := f(\tau(a)w^{-1}) = f(\tau(aw)).
\]

Then \( J : \mathbf{H}_s \to \mathbf{H}_s \) intertwines \( \pi_s \) and \( \pi_s \circ \tau \), and \( J^2 = 1 \). In our realization of \( \pi_s \) in \( L^2(\mathbb{R}) \) we have \( J(f)(x) = |x|^{-s-1}f(1/x) \), and
\[
A_s(J(g))(x) = \int_{-\infty}^{\infty} g(y)|1 - xy|^{s-1} dy.
\]

Hence
\[
\langle f | g \rangle_J = \langle f | A_s J g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y)|1 - xy|^{s-1} dy dx.
\]
Let $K_0$ be the completion of the space of smooth functions with compact support in $I = (-1, 1)$. Notice that the above inner product is defined on $C^c_s(I)$ for every $s$ as we only integrate over compact subsets of $(-1, 1)$.

The Bergman kernel for the domain $\{ z \in \mathbb{C} \mid |z| < 1 \}$ is $h(z, w) = 1 - \bar{z}w$, and it is well known (cf. [7, p. 268]) that $h(z, w)^{-\lambda}$ is a positive definite kernel function if and only if $\lambda \geq 0$. As our kernel is just $h(z, w)^{-(1-s)}$, restricted to the interval $I$, and $s < 1$, i.e., $1 - s > 0$, it follows that $\langle \cdot \mid \cdot \rangle_{J}$ is positive definite.

We also know (cf. [16]) that $S = H \exp C$ is a closed semigroup and that $\gamma I \subset I$, and actually $S$ is exactly the semigroup of elements in $\text{SL}(2, \mathbb{R})$ that act by contractions on $I$. Hence $S$ acts on $K$. By a theorem of Lüscher and Mack [15, 32], the representation of $S$ on $K$ extends to a representation of $G^c$, which in this case is the universal covering of $\text{SU}(1, 1)$ that is locally isomorphic to $\text{SL}(2, \mathbb{R})$. We notice that this defines a representation of $\text{SL}(2, \mathbb{R})$ if and only if certain integrality conditions hold: see [25].

We generalize this construction to the non-compactly causal symmetric spaces and in particular to the Cayley-type spaces. Furthermore we identify the resulting representation as an irreducible unitary highest weight representation of the dual group $G^c$. We restrict ourself to the case of characters induced from a maximal parabolic subgroup, which leads to highest weight modules with one-dimensional lowest $K^c$-type. This is meant as a simplification and not as a limitation of our method.

Assume now that $G$ is a semidirect product of $H$ and $N$ with $N$ normal and abelian. Define $\tau : G \to G$ by $\tau(hn) = h\bar{n}^{-1}$. Let $\pi \in \tilde{H}$ (the unitary dual) and extend $\pi$ to a unitary representation of $G$ by setting $\pi(hn) = \pi(h)$. In this case, $G^c$ is locally isomorphic to $G$, and $\pi$ gives rise to a unitary representation $\pi^c$ of $G^c$ by the formula $d\pi^c(X) = d\pi(X)$, $X \in \mathfrak{h}$, and $d\pi^c|_{\mathfrak{g}} = 0$. A special case of this is the 3-dimensional Heisenberg group, and the $(ax + b)$-group.

In sections 6 and 7, we show that, if we induce instead a character of the subgroup $N$ to $G$, then we have $(K_0/N)^{-1} = \{0\}$.

Our approach to the general representation correspondence $\pi \mapsto \pi^c$ is related to the integrability problem for representations of Lie groups (see [25]); but the present positivity viewpoint comes from Osterwalder-Schrader positivity: see [50, 51]. In addition the following other papers are relevant in this connection: [9, 22, 23, 27, 55, 59].

References


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