But $M$ is 3-connected so $|E(M) - X| \leq 1$ and it follows without difficulty that $M$ is a rank-3 wheel having $cl(A)$ as a triangle. Thus $(M, A)$ is not a minimal pair; a contradiction.

\begin{acks}

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\end{acks}

\begin{thebibliography}{19}

\bibitem{bixby82} Bixby, R.E., A simple theorem on 3-connectivity, \textit{Linear Algebra Appl.} \textbf{45} (1982), 123–126.


\bibitem{lemos97} Lemos, M. and Oxley, J., On the 3-connected matroids that are minimal having a fixed spanning restriction, submitted.

\bibitem{lemos00} Lemos, M. and Oxley, J., On size, circumference and circuit removal in 3-connected matroids, submitted.


\bibitem{oxley97} Oxley, J. and Wu, H., On the structure of 3-connected matroids and graphs, submitted.


\end{thebibliography}
since \( T_1 \) is a triangle of \( H \), and \( \{a_0, a_2\} \) and \( \{e_1, e_2\} \) are cocircuits of \( H \), we deduce that \( \{a_0, a_1, a_2\} \) is a triangle of \( H \); a contradiction. We conclude that Lemma 5.15 holds. \( \square \)

Now, for the minimal pair \((M', A')\), since \( X \cup Y \subseteq \{e_1, e_2\} \), we have \( \delta_E \leq 4 \).
Clearly \( \delta_A = 2 \). As \( M'[A'] = (M/\{a_0, a_2\})[A'] \) and \( \{a_0, a_1, a_2\} \) is a series class of \( M[A] \), we can obtain \( M[A] \) by taking the 2-sum of \( M'[A'] \) and a 4-circuit. Thus \( \delta_1 = 0, \delta_2 = 2, \delta_3 = 0, \) and \( \delta_\alpha = 0 \). Finally, it is clear that \( \delta_r \geq 0 \). Thus, by Lemma 5.4, \( \delta_m \geq 0 \). Hence, by (10),
\[
2 + 0 + 2 + 0 \leq \delta_A + \delta_1 + \delta_2 + \delta_m < \delta_\alpha + \delta_E \leq 0 + 4.
\]
This contradiction completes the proof of Proposition 5.2. \( \square \)

Theorem 5.1 is quite straightforward to prove by combining Proposition 5.2 and the following technical proposition that was proved in [6].

5.16. \textbf{Proposition.} Let \((M, A)\) be a minimal pair such that
\begin{enumerate}[(i)]  
\item \( M \) is not isomorphic to \( U_{1,3} \); and 
\item every element of \( E(M) - \text{cl}(A) \) belongs to some type-2 fan of length three in which the rim is contained in a 4-circuit of \( M[A] \) and the spokes are contained in \( E(M) - \text{cl}(A) \).
\end{enumerate}

Then
\[
|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - \beta(A, M),
\]
where
\[
\beta(A, M) = \begin{cases} 
1, & \text{when } A \text{ is a circuit of } M \text{ or } r(A) \neq r(M); \\
2, & \text{when } A \text{ is not a circuit of } M \text{ and } r(A) = r(M).
\end{cases}
\]

\textbf{Proof of Theorem 5.1.} Assume that the theorem fails and choose a minimal pair \((M, A)\) that is a counterexample and is chosen so that \((|E(M)|, -|A|)\) is lexicographically minimal. Suppose first that \( M \cong U_{1,3} \). Then \( |A| = 3 \) and it is straightforward to check that \((M, A)\) is not a counterexample to the theorem. Thus we may assume that \( M \not\cong U_{1,3} \). Moreover, by Proposition 5.2, \((M, A)\) satisfies hypothesis (ii) of Proposition 5.16. Thus the latter proposition implies that
\[
|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - \beta(A, M).
\]
As \(-\alpha(A, M) \geq -\beta(A, M) \) and \( \min\{\lambda_3(A, M), r(M) - r_M(A)\} \geq 0 \), it follows that
\[
|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M).
\]
But this is a contradiction since \((M, A)\) was chosen to be a counterexample to Theorem 5.1. \( \square \)

To conclude the paper, we now prove Theorem 1.4.

\textbf{Proof of Theorem 1.4.} We shall prove that \( A \) is spanning. The theorem will then follow from [6, Theorem 1.1]. Suppose that \( E(M) - \text{cl}(A) \) is non-empty. Then it contains a cocircuit \( D \) of \( M \). By Theorem 2.8, \( D \) must meet a triangle \( T \) of \( M \). If \( T \cap \text{cl}(A) \) is empty, then, by Lemma 4.1(viii), \( M \) has a triad that contains two elements of \( T \) and one of \( A \), so \( M[A] \) has a cooop; a contradiction. Thus \( T \) meets \( \text{cl}(A) \) so, by Lemma 4.1(ix), there is a chain \( T_1^*, \ldots, T_8^* \) whose rim is contained in \( \text{cl}(A) \). Since \( M[A] \) has no colloops, this rim is contained in a series class \( S \) of \( M[A] \). Since \( r(S) \leq 2 \), it follows that \( S \) is a triangle of \( M \). Thus if \( X = T_1^* \cup T \cup T_8^* \), then
\[
r(X) + r^*(X) - |X| \leq 3 + 3 - 5 = 1.
\]
$E(M'), W]$ is not a 2-separation of $M'$, it follows that $|Z \cap E(M')| \leq 1$. Hence $Z - \{a_0, a_2, e_1, e_2\} = \{a_1\}$, so $Z = F$.

Now $a_0, a_1,$ and $a_2$ are in series on $M \setminus \{e_1, e_2\}$ because $F$ is a fan. These elements are also in series in $M \setminus A$. Thus they are also in series in $M \setminus \operatorname{cl}(A)$. Hence $\operatorname{cl}(A)$ is spanned by $A - a_0$, so $\operatorname{cl}(A) - a_0$ is spanned by $A - a_0$. But $a_1$ and $a_2$ are cocircuits of $M \setminus \operatorname{cl}(A) - a_0$. Thus $A - R$ spans $\operatorname{cl}(A) - R$. Since $F = Z$, it follows that $A - R \subseteq W$. Hence $W$ spans $\operatorname{cl}(A) - R$. But $\operatorname{cl}(A) - R \subseteq \operatorname{cl}(A) \cap X$, so $W$ spans $\operatorname{cl}(A) \cap X$ and hence $\{Z, W \cup (X \cap \operatorname{cl}(A))\}$ is a $k$-separation of $M$, a contradiction. Thus (5.15.1) holds.

5.15.2. $|W| = 2$ and $W$ is a cocircuit of $H$.

Suppose that $|W| \geq 3$. Then a minor modification of the argument given in the last paragraph of the proof of Lemma 5.12 establishes that $F \subseteq Z$. This contradiction to (5.15.1) implies that $|W| \leq 2$. Then $W$ consists of either one or two cocircuits of $H$, or a 2-cocircuit of $H$.

Suppose that $W$ is a set of cocircuits of $H$. Then $W \subseteq \operatorname{cl}(A)$ since every element in $E(M) - \operatorname{cl}(A)$ belongs to a triangle of $H$ by Corollary 5.11. Orthogonality now implies that $W \subseteq A$. As $W$ is a set of cocircuits of $H$, it is a set of cocircuits of $M \setminus A$, so $W \cap R = \emptyset$. Hence $W$ is a set of cocircuits of $M'$. This contradiction completes the proof of (5.15.2).

5.15.3. $W = \{a_0, a_2\}$.

Since $W$ is a cocircuit of $M \setminus (X \cap \operatorname{cl}(A))$, it follows that $M$ has a cocircuit $D$ that contains $W$ and is contained in $W \cup (X \cap \operatorname{cl}(A))$. Hence $D \subseteq W \cup (\operatorname{cl}(A) - A)$. As $|D| \geq 3$, there is certainly an element of $\operatorname{cl}(A) - A$ in $D$.

Now $W$ does not contain a cocircuit of $M'$, so $W \cap (Y \cup \{a_0, a_2\}) \neq \emptyset$. Suppose that $Y \cap W \neq \emptyset$. Then, by Lemma 5.14, we may assume that $\{e_i\} \subseteq Y \subseteq \{e_1, e_2\}$. But $Y \neq \{e_1, e_2\}$ otherwise $a_1$ is a loop of $M \setminus Y$ and hence of $M'$. Thus $Y = \{e_1\}$. Now, by orthogonality, since $D$ meets $\operatorname{cl}(A) - A$, it must also meet $A$. But the only element of $D$ that can be in $A$ is the unique element of $W - Y$. Then, by orthogonality with $T_1$, this element must be $a_1$. Thus $|W \cap T_1| = 2$; a contradiction. We conclude that $Y \cap W = \emptyset$.

We may now assume that $a_0 \in W$. As $D \cap A \subseteq D \cap W$ and $D \cap A$ contains a cocircuit of $M \setminus A$, we deduce, since $R$ is a series class of $M \setminus A$, that $D \cap W = \{a_0, a_2\}$ and so (5.15.3) holds.

5.15.4. $\{e_1, e_2\}$ is a cocircuit of $H$.

Since $F$ is a fan of the 3-connected matroid $M$, the set $\{a_0, e_1, e_2, a_2\}$ is a cocircuit of $M$. Therefore this set is a union of cocircuits of $H$. By (5.15.2) and (5.15.3), $\{a_0, a_2\}$ is a cocircuit of $H$. Moreover, $\{e_1, a_1, e_2\}$ is a triangle of $H$. Assume that $\{e_1, e_2\}$ is not a cocircuit of $H$. Then, by orthogonality, $\{a_0, e_1, e_2\}$ or $\{a_2, e_1, e_2\}$ is a cocircuit of $H$. Thus $H \setminus A$, which equals $M \setminus A$, has $\{a_0\}$ or $\{a_2\}$ as a cocircuit. This contradiction finishes the proof of (5.15.4).

By (5.15.4), $\{e_1, e_2\}, E(H) - \{e_1, e_2\}$ is a 2-separation of $H$. Thus, as $|\{e_1, e_2\} \cap T_1| \geq 2$, the triangle $T_1$ implies that $\{e_1, e_2, a_1\}, E(H) - \{e_1, e_2, a_1\}$ is a 2-separation of $H$. Moreover, for each $i$ in $\{0, 2\}$, the triad $T_i$ of $M$ implies that $\{e_1, e_2, a_1, a_i\}, E(H) - \{e_1, e_2, a_1, a_i\}$ is also a 2-separation of $H$ unless $E(H) = F$. Assume that $E(H) \neq F$. Then $H$ has a 2-separation $\{Z, W\}$ such that $Z \geq \{e_1, e_2, a_1, a_0\}$ and $|Z \cap F|$ is maximum subject to this. Then (5.15.3) implies that $W = \{a_0, a_2\}$; a contradiction. We conclude that $E(H) = F$. Then,
which, as above, must equal \( \{X, Y\} \). Thus \( T_0 \subseteq X \) and, by symmetry, \( T_2 \subseteq X \). We conclude that \( F \subseteq X \), so \( X \) spans \( y \). This contradiction completes the proof that 
\[ y \in A. \]

We may now assume that \( F \) has no chord, otherwise Proposition 5.2 follows.

5.13. Lemma. \( M/\{a_0, a_2\} \) is 3-connected.

Proof. As \( a_0 \in T_0 - T_1 \) and \( T_0, T_1, T_2 \) is a type-2 fan, Lemma 2.1 implies that \( a_0 \) is non-essential. Thus \( M/a_0 \) is 3-connected. We show next that \( M/a_0 \) has no triangle containing \( a_2 \). Assume that such a triangle \( T' \) exists.

Suppose first that \( e_2 \in T' \). Then \( T' = \{a_2, e_2, e_3\} \) for some \( e_3 \in E(M) - \{a_0, a_2, e_2\} \). Since \( e_2 \not\in cl(A) \), it follows that \( e_3 \not\in cl(A) \). Thus either \( T' \) or \( T' \cup a_0 \) is a circuit of \( M \). The first possibility contradicts the fact that \( T_0, T_1, T_2 \) is a fan; the second contradicts orthogonality with the triad \( T_0 \) since \( e_3 \neq e_1 \) otherwise \( M/a_0 \) has a line containing \( \{a_1, a_2, e_1, e_2\} \).

We may now assume that \( e_2 \not\in T' \). Then, by orthogonality with \( T_2 \), we have \( T' = \{a_2, a_1, g\} \) for some element \( g \). By Lemma 2.9, \( T' \) is not a triangle of \( M \) so \( T' \cup a_0 \) is a circuit of \( M \) and hence \( g \) is a chord of \( M/a_1 \); a contradiction.

We conclude that there is indeed no triangle of \( M/a_0 \) containing \( a_2 \). But the cosimplification of \( M/a_0 \) is not 3-connected, so the simplification of \( M/a_0/a_2 \), which equals \( M/\{a_0, a_2\} \), is 3-connected.

Now let \( A' = A - \{a_0, a_2\} \) and let \( M' \) be a 3-connected matroid such that \( M'|A' = (M/\{a_0, a_2\})|A' \) and \( (M', A') \) is a minimal pair. Then \( M' = (M/\{a_0, a_2\}) \setminus X/Y \) for some disjoint subsets \( X \) and \( Y \) of \( E(M) - \{a_0, a_2\} \).

5.14. Lemma. \( Y \cup (X - cl(A)) \subseteq \{e_1, e_2\} \).

Proof. Suppose that there is an element \( z \) in \( [Y \cup (X - cl(A))] - \{e_1, e_2\} \). We shall first show that \( z \not\in cl(A) \). If not, then \( z \in Y \cap cl(A) \) and \( r(M'|A') \leq r((M/\{a_0, a_2, z\})|A') = r((M/\{a_0, a_2\})|A') - 1 \); a contradiction. We conclude that we do indeed have that \( z \not\in cl(A) \). Thus \( z \in E(M) - (cl(A) \cup F) \). Then, by Corollary 5.11, \( z \) is in a type-2 fan \( F' \) of \( M \) of length three whose rim is a series class of \( M/A \) and such that \( F' \subseteq E(M) - F \). Since \( z \not\in cl(A) \), it follows that \( z \) is a spoke of \( F' \). Therefore \( M/z \) has a 2-cocircuit contained in \( A - \{a_0, a_2\} \) that remains a 2-cocircuit of \( M/\{a_0, a_2\} \). Now the last matroid is 3-connected of size at least five since \( |A| \geq 7 \). Hence \( z \not\in X \). On the other hand, if \( z \in Y \), then \( M/z \) has a 2-circuit containing a spoke \( z' \) of \( F' \). Thus \( M' \) is a minor of \( M/z \) and hence of \( M/z' \). Again we obtain a contradiction.

5.15. Lemma. \( X \cap cl(A) = \emptyset. \)

Proof. Suppose that \( X \cap cl(A) \neq \emptyset. \) Let \( H = M \setminus (X \cap cl(A)) \). Then \( H|A = M|A. \)

As \( (M, A) \) is a minimal pair; it follows that \( H \) is not 3-connected. For some \( k \) in \( \{1, 2\} \), let \( \{Z, W\} \) be a \( k \)-separation of \( H \) for which \( |Z \cap T_1| \geq 2 \) and \( |Z \cap F| \) is maximum subject to this condition.

5.15.1. \( F \not\subseteq Z. \)

Assume the contrary. Then \( |Z| \geq 5 \). But, by Lemma 5.14, \( E(H) - E(M') \subseteq \{a_0, a_2, e_1, e_2\} \subseteq F \). Thus \( Z \cap E(M') \supseteq Z - \{a_0, a_2, e_1, e_2\} \supseteq \{a_1\} \). Since \( Z \cap
As each series class of $M'|A'$ is contained in a series class of $M|A$, we have $\delta_3 \geq 0$. Moreover, as $\delta_3$ is clearly zero, Lemma 5.4 implies that

$$\delta_m \geq 0.$$ 

Therefore, by (10), we obtain a contradiction unless

$$n + 0 + n + 0 \leq \delta_A + \delta_1 + \delta_2 + \delta_m < \delta_E + \delta_\alpha = (n + 1) + \delta_\alpha.$$ 

Thus $1 \leq n - 1 < \delta_\alpha \leq 1$; a contradiction. \hfill \Box

On combining Lemmas 5.8 and 5.10, we obtain the following.

5.11. Corollary. If $f \in E(M) - \text{cl}(A)$, then $f$ is in a type-2 fan of length three in which the rim is a series class of $M|A$ and the spokes are contained in $E(M) - \text{cl}(A)$. Moreover, two distinct such fans are disjoint.

Proof. Either $f$ is in a triangle of $M$ or not. In the first case, by Lemma 5.10, $f$ is in a unique triad but no triangle and the first part of the corollary follows by Lemma 5.8. In the second case, by Lemma 5.5, $f$ is in a triangle with some element of $\text{cl}(A)$. Then, by Lemma 4.1(ix), $f$ satisfies (i) of Lemma 5.8 and again the first part of the corollary follows by that lemma.

Now observe that the intersection of two different fans $F_1$ and $F_2$ satisfying the specified conditions is a subset of the intersection of their rims or the intersection of their sets of spokes. But each rim is a series class of $M|A$ so these sets are either equal or disjoint. Thus if $F_1$ meets $F_2$, then $F_1$ and $F_2$ have a common element that is essential in $M$. But this violates Theorem 2.2. \hfill \Box

Now let $T_0, T_1, T_2$ be a type-2 fan $F$ of $M$ where $T_0 = \{a_0, e_1, a_1\}, T_1 = \{e_1, a_1, e_2\}$, and $T_2 = \{a_1, e_2, a_2\}$ where $\{a_0, a_1, a_2\} \subseteq \text{cl}(A)$ and $\{e_1, e_2\} \cap \text{cl}(A) = \emptyset$. A chord of $F$ is an element $y$ such that $R \cup y$ is a circuit of $M$ where $R$ is the rim $\{a_0, a_1, a_2\}$ of $F$. In view of the last corollary, to complete the proof of Proposition 5.2, it suffices to show that $F$ has a chord that is contained in $A$.

5.12. Lemma. If $y$ is a chord of $F$, then $y$ is unique and $y \in A$.

Proof. If $\{y, a_0, a_1, a_2\}$ and $\{y', a_0, a_1, a_2\}$ are circuits of $M$, then $\{y, y', a_0, a_2\}$ contains a circuit, which, by orthogonality, must be $\{y, y'\}$; a contradiction. Thus $y$ is unique. Now assume that $y \notin A$. Then $M\backslash y$ is not 3-connected. Let $\{X, Y\}$ be a 2-separation of $M\backslash y$ and suppose, without loss of generality, that $|X \cap T_1| \geq 2$. Subject to this restriction, choose the 2-separation $\{X, Y\}$ such that $|X \cap F|$ is maximum.

We now show that $|Y| \geq 3$. Assume that $|Y| = 2$. As $M$ is simple, $Y$ must be a cocircuit of $M\backslash y$, and so $Y \cup y$ is a triad $T^*$ of $M$. By orthogonality, $T^* \cap R \neq \emptyset$. By Lemma 2.9, $a_1 \notin T^*$. Hence we may assume that $a_0 \in T^*$. Because $R$ is a series class of $M|A$, the unique element, $z$ say, of $T^* - \{y, a_0\}$ must belong to $A$ otherwise $a_0$ is a coloop of $M|A$. Thus $z$ and $a_0$ are in series in $M|A$ so $z = a_2$, and $T^* = \{y, a_0, a_2\}$. By elimination and orthogonality, $\{a_0, e_1, e_2, a_2\}$ is a cocircuit of $M$. Hence $\{\{a_0, e_1, e_2, a_2\} \cup \{y, a_0, a_2\}\} - a_0$ contains a cocircuit $D$ of $M$. But $M|A$ has a circuit containing $\{a_0, a_1, a_2\}$ so, by orthogonality, $D = \{e_1, e_2, y\}$. This is a contradiction since $|D \cap (R \cup y)| = 1$. We conclude that $|Y| \geq 3$.

Next we observe that, since $X$ spans $T_1$, it follows that $\{X \cup T_1, Y - T_1\}$ is a $2$-separation of $M\backslash y$. By the choice of $\{X, Y\}$, we deduce that $X \cup T_1 = X$, so $T_1 \subseteq X$. Thus $X$ spans $T_0$ in $M^*$, so $\{X \cup T_0, Y - T_0\}$ is a $2$-separation of $M\backslash y$,
Since $\delta_r \in \{0, 1\}$, it follows, by Lemma 5.4, that
\[ \delta_m \geq 0. \]
Thus, by (10),
\[ 1 + 0 + 1 + 0 - 0 \leq 1 + 0 + 1 + \delta_m - 0 = \delta_A + \delta_1 + \delta_2 + \delta_m - \delta_3 < \delta_E. \]
Since $\delta_E \leq 3$, we deduce that $\delta_E = 3$ and $\delta_m = 0$. Hence (b) holds, that is, $M' = M/\{a\backslash e/e', e, e_1, e_2\}$, so $\delta_r = 1$. Thus, by Lemma 5.4, $\delta_3 \neq 1$ otherwise $\delta_m = 1$. Hence $\delta_3 = 0$ so $|S_a| \leq 3$. Therefore, by (b), $S_a = \{a_1, a_2, a_3\}$ and the fan
\[ \{a, e, a_1\}, \{e, e_1, e_2\}, \{a_1, e_1, a_2\} \]
produces all the conditions asserted in the lemma. \qed

5.9. Lemma. $|A| \geq 7$.

Proof. As $E(M) - \text{cl}(A) \neq \emptyset$, there is a cocircuit $D$ of $M$ that avoids $\text{cl}(A)$. Moreover, since $(M, A)$ is a minimal pair, for all elements $e$ of $D$, the matroid $M/e$ is not 3-connected. Thus, by Theorem 2.8, $M$ has two triangles $T_1$ and $T_2$ meeting $D$ in distinct subsets. By Lemma 5.5, each $T_i$ meets $\text{cl}(A)$. Hence, by Lemma 4.1(ix), there is a unique element $a_{T_i}$ in $T_i \cap \text{cl}(A)$ and there is a chain $T_{1T_i}, T_i, T_{2T_i}$ whose rim $A_{T_i}$ is contained in $\text{cl}(A)$. Take an element $f_1$ of $(T_1 - T_2) - \text{cl}(A)$. By Lemma 5.8, $f_1$ is in a 5-element fan $F_1'$ whose rim is a series class of $M/A$ and all of whose spokes are in $E(M) - \text{cl}(A)$. Clearly $T_2 - \text{cl}(A)$ has an element $f_2$ that is not in $F_1'$ and, by Lemma 5.8 again, $f_2$ is in a 5-element fan $F_2'$ whose rim is a series class of $M/A$.

The rims $R_1$ and $R_2$ of $F_1'$ and $F_2'$ are both series classes of $M/A$ and, by Theorem 2.2, these rims are not equal. Hence $R_1$ and $R_2$ are disjoint and so $|A| \geq 6$. We may now assume that $|A| = 6$ otherwise the lemma holds. But, in that case, $M/A$ is the union of the two series classes $R_1$ and $R_2$. Hence each $R_i$ is a triangle of $M/A$. This is a contradiction as $F_1'$ is a fan. \qed

5.10. Lemma. An element $e$ of $E(M) - \text{cl}(A)$ that does not belong to a triangle of $M$ belongs to exactly one triad.

Proof. The element $e$ certainly belongs to at least one triad, otherwise, by Lemma 2.3, either $M/e$ or $M/e$ is 3-connected; a contradiction to the fact that $(M, A)$ is a minimal pair. Now let $T_1^*, T_2^*, \ldots, T_n^*$ be the triads of $M$ that contain $e$ and suppose that $n \geq 2$. For all $i$, we must have that $|T_i^* - \text{cl}(A)| < 2$, otherwise Tutte’s triangle lemma (2.4) implies that $e$ is in a triangle. Hence $|T_i^* \cap \text{cl}(A)| \geq 2$ for all $i$ and it follows, by Lemma 5.6, that $|T_i^* \cap A| = 2$. Let $T_i^* \cap A = \{a_1, b_i\}$ for all $i$. Then since, by the previous lemma, $|A| \geq 7$, Lemma 4.7 implies that $(M', A')$ is a minimal pair where $M' = M/e/\{a_1, a_2, \ldots, a_n\}$ and $A' = A - \{a_1, a_2, \ldots, a_n\}$.

Clearly
\[ \delta_A = n \quad \text{and} \quad \delta_E = n + 1. \]
Moreover, for each $i$, either $a_i$ is in series with $b_i$ in $M/A$, or $a_i$ is a coloop of $M/A$. Without loss of generality, assume that the set of coloops of $M/A$ in $\{a_1, a_2, \ldots, a_n\}$ is $\{a_1, a_2, \ldots, a_k\}$. Then a matroid isomorphic to $M/A$ can be obtained from $M'/A'$ in two stages: first take the direct sum of $M'/A'$ with the $k$ coloops $a_1, a_2, \ldots, a_k$; and then take the 2-sum of the resulting matroid with $n - k$ copies of $U_{2,3}$ using $b_{k+1}, b_{k+2}, \ldots, b_n$ as basepoints. Thus
\[ \delta_1 = k \geq 0 \quad \text{and} \quad \delta_2 = n. \]
the fan $F$ and the fans $\{a,e,a_1\}, \{e,a_1,e'\}, \{a_1,e',a_2\}$ whose existence is asserted in Lemma 4.3(iii). By Theorem 2.2, if these fans are distinct, then $F$ has five elements and has $e$ as a rim element; a contradiction. Hence these two fans are the same, so $f \in \{e,e'\}$. We conclude that if the cosimplification of $M\setminus e$ is isomorphic to $M\setminus e/a$, then (a) or (b) holds.

Now assume that the cosimplification of $M\setminus e$ is not isomorphic to $M\setminus e/a$. Then $M$ has a triad $T^*_1$ that contains $e$ and is different from $T^*$. It follows, using orthogonality, that $T^*_1$ must contain the first two spokes, $e$ and $e'$, of $F$. Moreover, by Lemma 4.1(v), the other element $a'$ of $T^*_1$ is in $A$. By orthogonality again, $F$ cannot have a third spoke. Hence $e$ and $e'$ are the only two spokes of $F$, so $f \notin \{e,e'\}$. Moreover, $T^*$ and $T^*_1$ are the only triads of $M$ containing $e$, and the cosimplification of $M\setminus e$ is isomorphic to $M\setminus e/a/e'$. We show next that

$$(M\setminus e/a/e')(A-a) = (M/a)(A-a).$$

This follows if $(M/a)((A-a) \cup e')$ has $e'$ as a coloop. Assume the contrary. Then $e'$ is in the closure of $A-a$ in $M/a$, so $e'$ is in the closure of $A$ in $M$; a contradiction. Clearly $M\setminus e/a/e'$ has a minimal 3-connected minor $M'$ for which $M'(A-a) = (M/a)(A-a)$. Thus, by Lemma 4.3, we have that (a) or (b), and hence (b), holds.

We now know that (a) or (b) holds, and we examine the minimal pair $(M', A')$. Since $|E(M')| < |E(M)|$, Theorem 5.1 holds for $(M', A')$. Clearly

$$\delta_A = 1.$$ 

Moreover,

$$\delta_e = \begin{cases} 0, & \text{when } M' = M/a; \\ 1, & \text{when } M' = M/a\setminus e/e'. \end{cases}$$

The rest of the argument will be broken into the following cases.

(I) $a$ is a coloop of $M/A$;  
(II) $a$ is in a series class of $M/A$.

In case (I), since $M'|A' = (M/a)|A'$,

$$\delta_1 = 1, \delta_2 = 1, \text{ and } \delta_3 = 0.$$ 

Thus, by Lemma 5.4, $\delta_m \geq 0$. Hence, by (10),

$$1 + 1 + 1 + 0 \leq \delta_A + \delta_1 + \delta_2 + \delta_m < \delta_\alpha + \delta_E.$$ 

Thus, as $\delta_\alpha \leq 1$ and $\delta_E \leq 3$, we deduce that $\delta_\alpha = 1$ and $\delta_E = 3$. The first of these implies that $A$ is not a circuit of $M$ but $A-a$ is a circuit of $M'$; and the second implies that (b) holds. By (b), $M\setminus e$ has $\{a,a_1\}$ as a cocircuit. Thus $M/A$ has $\{a,a_1\}$ as a union of cocircuits. But $a$ is a coloop of $M/A$. Hence so is $a_1$. This contradicts the fact that $A-a$ is a circuit of $(M/A)/a$. Thus (I) cannot occur.

Now consider (II). In this case, $M/A$ is obtained from $M'|A'$ by adding $a$ in series to some element of $A'$. Thus $M/A$ is isomorphic to the 2-sum of $M'|A'$ and a copy of $U_{2,3}$. Hence

$$\delta_1 = 0, \delta_2 = 1, \text{ and } \delta_3 = 0.$$ 

Moreover, if $S_a$ is the series class of $M/A$ containing $a$, then

$$\delta_3 = \begin{cases} 0, & \text{when } |S_a| \leq 3; \\ 1, & \text{when } |S_a| \geq 4. \end{cases}$$
Proof. We shall show first that, in each case, there are elements \( a \) of \( A \) and \( e \) of \( E(M) - \text{cl}(A) \) such that \( M/a\{e \} \) has a minimal 3-connected minor \( M' \) for which \( M'\{A\} = (M/a)|A' \) where \( A' = A - a \) and either

(a) \( M' = M/a\{e \} \) where \( \{a, e\} \) is contained in a triad of \( M \); or

(b) \( M' = M/a\{e\{e' \} \) for some element \( e' \) of \( E(M) - \text{cl}(A) \) such that \( f \in \{e, e'\} \).

Moreover, for some elements \( a_1 \) and \( a_2 \) of \( A \), the set \( \{e, a_1, e' \} \) is a triangle, and \( \{a, e, a_1\}, \{e, a_1, e' \}, \{a_1, e', a_2\} \) is a type-2 fan of \( M \).

Suppose that (ii) occurs and let \( T^* \) be the unique triad containing \( f \). Then \( T^* - f \) meets \( A \), otherwise, by Lemma 4.1(v), \( |T^* - \text{cl}(A)| \leq 1 \) in which case, by Lemma 5.6, \( |T^* \cap A| \geq 2 \); a contradiction. Let \( T^* - f = \{a, g\} \) where \( a \in A \). Now suppose that \( a \) is in a triangle of \( M \). Then, since this triangle cannot contain \( f \), it must contain \( g \). But, in that case, the cosimplification of \( M\{f \} \) is not 3-connected, contradicting Lemma 4.1(i). We conclude that \( a \) is in no triangles of \( M \).

We are aiming to apply Lemma 4.3 to \( M \). We know, by Lemmas 5.5 and 5.6, that hypotheses (a) and (b) of that lemma hold and that \( a \) is in no triangles of \( M \). Now consider \( M/a \). It must be 3-connected otherwise neither \( M/a \) nor \( M/f \) is 3-connected and the dual of Tutte's triangle lemma (2.4) implies that \( a \) is in a triangle; a contradiction. Next consider \( M/f \). Since \( f \) is in no triangles, the simplification of \( M/f \) equals \( M/f \) and so is not 3-connected. Thus the cosimplification of \( M\{f \} \) is 3-connected. Since \( T \) is the unique triad of \( M \) containing \( f \), this cosimplification is \( M\{f \}/a \). Thus \( M/a\{f \} \) is 3-connected. Therefore there is a minimal 3-connected minor \( M' \) of \( M/a \) for which \( M'\{A - a\} = (M/a)|(A - a) \) such that \( M' \) is a minor of \( M/a\{f \} \). Then Lemma 4.3 and the fact that \( f \) is in no triangles imply that \( M' = (M/a)\{f \} \). We conclude that if (ii) occurs, then (a) holds with \( e = f \).

Next suppose that (i) holds, and take a maximal such chain \( T_1, T_2, \ldots, T_k \) of \( M \). Then \( k \geq 3 \). Assume that there is a set \( T_{k+1} \) such that \( T_2, T_3, \ldots, T_{k+1} \) is a chain. Let \( T_i = \{e_i, e_{i+1}, e_{i+2}\} \) for all \( i \) in \( \{1, 2, \ldots, k + 1\} \). Suppose first that \( T_1, T_2, \ldots, T_{k+1} \) is a chain. By the choice of \( T_1, T_2, \ldots, T_k \), we deduce that either \( e_{k+3} \) is a spoke of \( T_1, T_2, \ldots, T_{k+1} \) that is in \( \text{cl}(A) \), or \( e_{k+3} \) is a rim element of \( T_1, T_2, \ldots, T_{k+1} \) that is not in \( \text{cl}(A) \). In the first case, \( T_{k+1} \) is a triangle containing two elements of \( \text{cl}(A) \), so \( e_{k+1} \in \text{cl}(A) \); a contradiction. In the second case, by Lemmas 5.5 and 4.1(xii), \( e_{k+2} \) belongs to a triad of \( T_0 \) that meets \( \text{cl}(A) \) in two elements. By orthogonality with \( T_k \), we must have that \( e_{k+1} \in T_0 \), so, by Lemma 2.9, \( T_0 = T_{k+1} \); a contradiction. We conclude that \( T_1, T_2, \ldots, T_{k+1} \) is not a chain. Then \( e_{k+3} \in T_1 \cup T_2 \cup \ldots \cup T_k \) and it follows, by orthogonality, that \( e_{k+3} = e_1 \) and that \( T_1 \) and \( T_{k+1} \) are either both triangles or are both triads. Then, by Lemma 5.7, it follows that \( M \) is isomorphic to a wheel or whirl of rank \( k + 2 \). Moreover, clearly this wheel or whirl has rim \( \text{cl}(A) \) and so we have a contradiction to Lemma 5.3. We conclude that there is no set \( T_{k+1} \) such that \( T_2, T_3, \ldots, T_{k+1} \) is a chain. Similarly, there is no set \( T_0 \) such that \( T_0, T_1, \ldots, T_{k-1} \) is a chain. We conclude that \( T_1, T_2, \ldots, T_k \) is a fan \( F \). Moreover, since \( e_1 \) and \( e_{k+2} \) are non-essential, it follows by Lemma 4.1(ix) that both \( T_1 \) and \( T_k \) are triads of \( M \).

Let \( \{a, e, a_1\} \) be the first link \( T^* \) of \( F \) where \( a \) is the rim element of \( T^* \) that is in no triangles, and \( e \) is the spoke of \( T^* \), the first spoke of \( F \). By Lemma 5.6, \( \{a, e_1\} \subseteq A \). By Lemma 4.1(i), the cosimplification of \( M\{e \} \) is 3-connected. Suppose that this cosimplification is isomorphic to \( M\{e/a \} \), and let \( M' \) be a minimal 3-connected minor of \( M\{e/a \} \) for which \( M'\{A - a\} = (M/a)|(A - a) \). Then Lemma 4.3 implies that (a) or (b) holds provided that, as we now show, \( f \in \{e, e'\} \). Clearly \( e \) is in both
Proof. Suppose that $|T^* \cap A| \leq 1$. By Lemma 2.6, $T^* \cap A = \{a\}$ for some $a$ in $A$. Choose $e$ in $(T^* \cap \text{cl}(A)) - A$. Let $M' = M$ and $A' = A \cup e$. Clearly $(M', A')$ is a minimal pair. Observe that the connected component $N_e$ of $M|(A \cup e)$ that contains $e$ must also contain $a$. Moreover, $a$ is a coloop of $N_e\backslash e$. Let $l$ be the number of coloops of $N_e\backslash e$ that are not coloops of $N_e$. Then $l \geq 1$. By Lemma 2.9(i) of [6], we have that
\[ \delta_1 + \delta_2 \geq l. \]
Observe also that
\[ \delta_A = -1, \delta_E = 0, \text{ and } \delta_r = 0, \]
where the last of these holds because $A$ spans $e$.

Now consider a series class $S$ of $M|A$. Its contribution to $\lambda_3(A, M)$ is $\max\{0, |S| - 3\}$. In $M|(A \cup e)$, there is a partition $\{S_1, S_2, \ldots, S_k\}$ of $S$ such that each $S_i$ is a series class of $M|(A \cup e)$. The total contribution of these series classes to $\lambda_3(A \cup e, M)$ is
\[ \sum_{i=1}^k \max\{0, |S_i| - 3\}, \]
which clearly does not exceed $\max\{0, |S| - 3\}$. The only non-trivial series class of $M|(A \cup e)$ that is not contained in a non-trivial series class of $M|A$ is the one that contains $e$ and the set of $l$ elements that are coloops of $N_e\backslash e$ but not of $N_e$. Its contribution to $\lambda_3(A \cup e, M)$ is $\max\{0, l - 2\}$. We conclude that
\[ \delta_3 \geq -\max\{0, l - 2\} = \min\{0, 2 - l\} \]
and, since $\delta_r = 0$, Lemma 5.4 implies that
\[ \delta_m \geq \min\{0, 2 - l\}. \]
 Thus
\[ \delta_A + (\delta_1 + \delta_2) + \delta_m - \delta_E \geq -1 + l + \min\{0, 2 - l\} - 0. \]
By (10), we must have that $\delta_A > l - 1 + \min\{0, 2 - l\}$. Hence
\[ \delta_A > \min\{l - 1, 1\}. \]
But $\delta_A \leq 1$ and $l \geq 1$, so $\delta_A = 1$ and $l - 1 = 0$. The first of these implies that $A \cup e$ is a circuit of $M$. Hence $N_e = M|(A \cup e)$ and $A$ equals the set of coloops of $N_e\backslash e$ that are not coloops of $N_e$. Thus $l = |A| \geq 3$, by (9); a contradiction. \hfill \Box

The proof of Lemma 5.8 will use the following result of Oxley and Wu [11, Lemma 2.4].

5.7. Lemma. For some $n \geq 2$, let $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ be disjoint subsets of the ground set of a 3-connected matroid $N$. Suppose that, for all $i$ in $\{1, 2, \ldots, n - 1\}$ and all $j$ in $\{1, 2, \ldots, n\}$, $\{e_i, f_i, e_{i+1}\}$ is a triangle and $\{f_j, e_{j+1}, f_{j+1}\}$ is a triad, where all subscripts are read modulo $n$. Then $N$ is isomorphic to a wheel or whirl of rank $n$.

5.8. Lemma. Let $f$ be an element of $E(M) - \text{cl}(A)$. Suppose that either
\begin{enumerate}
\item[(i)] $f$ is in a chain with at least three links such that all the spokes of this chain are in $E(M) - \text{cl}(A)$ and the rim of this chain is contained in $\text{cl}(A)$; or
\item[(ii)] $f$ is in a unique triad of $M$ but is in no triangles.
\end{enumerate}
Then $f$ is a type-2 fan of length three in which the rim is a series class of $M|A$ and the spokes are contained in $E(M) - \text{cl}(A)$.
On taking the difference of these two inequalities, we get

\[ \delta_A + \delta_1 + \delta_2 + \delta_m - \delta_\alpha - \delta_E < 0, \]

where

\[ \delta_A = |A| - |A'|, \]
\[ \delta_1 = \lambda_1(A, M) - \lambda_1(A', M'), \]
\[ \delta_2 = \lambda_2(A, M) - \lambda_2(A', M'), \]
\[ \delta_m = \min \{ \lambda_3(A, M), r(M) - r_M(A) \} - \min \{ \lambda_3(A', M'), r(M') - r_{M'}(A') \}, \]
\[ \delta_\alpha = \alpha(A, M) - \alpha(A', M'), \]
\[ \delta_E = |E(M)| - |E(M')|. \]

Hence we shall arrive at a contradiction whenever (10) is not satisfied. To deal with \( \delta_m \), we define two other differences:

\[ \delta_3 = \lambda_3(A, M) - \lambda_3(A', M'), \]
\[ \delta_r = [r(M) - r_M(A)] - [r(M') - r_{M'}(A')]. \]

The elementary proof of the next lemma is omitted.

5.4. Lemma. (i) If \( \delta_r \) or \( \delta_3 \) is zero, then \( \delta_m \geq \min \{ \delta_r, \delta_3 \} \).

(ii) If \( \delta_r = \delta_3 \), then \( \delta_m = \delta_r = \delta_3 \).

5.5. Lemma. Every triangle of \( M \) that meets \( E(M) - \text{cl}(A) \) contains an element of \( \text{cl}(A) \).

Proof. Suppose that \( T \) is a triangle of \( M \) that avoids \( \text{cl}(A) \). By (viii) of Lemma 4.1, there is a triad that meets \( T \) in exactly two elements and contains some element \( a \) of \( A \). Then, by Lemma 4.2, if \( M' = M/T \) and \( A' = A - a \), then \( (M', A') \) is a minimal pair and

\[ (M/T)|(A - a) = M|(A - a). \]

As \( |E(M')| < |E(M)| \), the minimal pair \( (M', A') \) satisfies Theorem 5.1. Hence (10) holds. We shall now consider each individual difference in (10). Clearly

\[ \delta_A = 1 \text{ and } \delta_E = 3. \]

Next we note that, since \( (M/T)|(A - a) = M|(A - a) \) and \( a \) is a coloop of \( M|A \), we have

\[ \delta_1 = \delta_2 = 1 \text{ and } \delta_3 = 0. \]

Now \( r(M') = r(M) - 2 \) and \( r_{M'}(A') = r_M(A - a) = r_M(A) - 1 \). It follows that \( \delta_r = 1 \) and hence, by Lemma 5.4,

\[ \delta_m \geq 0. \]

Finally, (viii) of Lemma 4.1 implies that \( M|A \) has at least three coloops including \( a \), and therefore

\[ \delta_\alpha = 0. \]

On combining the above differences, we have

\[ \delta_A + \delta_1 + \delta_2 + \delta_m - \delta_\alpha - \delta_E \geq 1 + 1 + 1 + 0 - 0 - 3 = 0. \]

This contradiction to (10) completes the proof of the lemma.

5.6. Lemma. If \( T^* \) is a triad of \( M \) such that \( |T^* \cap \text{cl}(A)| \geq 2 \), then \( |T^* \cap A| \geq 2. \)
5.2. Proposition. Let \((M, A)\) be a minimal pair that is a counterexample to Theorem 5.1 and is chosen so that
\[
(|E(M)|, -|A|)
\]
is lexicographically minimal among such counterexamples. Then every element of \(E(M) - \text{cl}(A)\) belongs to some type-2 fan of length three in which the rim is contained in a 4-circuit of \(M\setminus A\) and the spokes are contained in \(E(M) - \text{cl}(A)\).

Proof. Suppose that \((M, A)\) does not satisfy the proposition. Then, clearly,
\[
E(M) - \text{cl}(A) \neq \emptyset.
\] (8)
Moreover, by (iii) of Lemma 4.1,
\[
|A| \geq 3.
\] (9)
The rest of the proof of the proposition will be broken up into a sequence of lemmas, the first of which is as follows.

5.3. Lemma. \(M\) is neither a wheel having rim \(\text{cl}(A)\) nor a whirl having rim \(A\).

Proof. Suppose that \(M\) is the rank-\(n\) whirl having rim \(A\). Then \(A\) is independent and
\[
|A| + \lambda_1(A, M) + \lambda_2(A, M)
+ \min \{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M) = 3|A| - 1 = 3n - 1.
\]
But \((M, A)\) is a counterexample to Theorem 5.1 and \(|E(M)| = 2n\), so \(2n > 3n - 1\); a contradiction.

Next suppose that \(M\) is the rank-\(n\) wheel having rim \(\text{cl}(A)\) but \(A \neq \text{cl}(A)\). Then \(A = \text{cl}(A) - a\) for some element \(a\), so \(A\) is independent. Again, we have that
\[
|A| + \lambda_1(A, M) + \lambda_2(A, M)
+ \min \{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M) = 3|A| - 1 = 3n - 4.
\]
But \(|E(M)| = 2n\) and \((M, A)\) is a counterexample to Theorem 5.1. Thus \(2n > 3n - 4\) so \(n \leq 3\). Hence \(|A| = n - 1 \leq 2\); a contradiction to (9).

Finally, suppose that \(M\) is the rank-\(n\) wheel and \(A\) is its rim. In this case, \(n \geq 4\), otherwise \(n = 3\) and \(A\) is a 3-element circuit so \((M, A)\) is a minimal pair; a contradiction. Now \(\lambda_1(A, M) = 1, \lambda_2(A, M) = n - 2, \lambda_3(A, M) = n - 3, r(M) - r(A) = 1\), and \(\alpha(A, M) = 0\). Thus
\[
2n = |E(M)| > n + 1 + (n - 2) + 1 - 0.
\]
This contradiction completes the proof of Lemma 5.3.

In several of the lemmas below, we shall replace the minimal pair \((M, A)\) by another minimal pair \((M', A')\) such that \(|E(M'), \text{cl}(A')|\) is lexicographically less than \(|E(M), \text{cl}(A)|\). Thus \((M', A')\) must satisfy the conclusion of Theorem 5.1 although \((M, A)\) does not. Hence
\[
|A| + \lambda_1(A, M) + \lambda_2(A, M)
+ \min \{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M) - |E(M)| < 0
\]
and
\[
|A'| + \lambda_1(A', M') + \lambda_2(A', M')
+ \min \{\lambda_3(A', M'), r(M') - r_M'(A')\} - \alpha(A', M') - |E(M')| \geq 0.
\]
Suppose that this fails for some $i$. Then, since $T_i^*$ is a triad of $M$ and $T_i^* - e$ is a cocircuit of $(M\setminus X/Y)\setminus e$, it follows that either (i) $T_i^*$ is contained in a series class of $M\setminus X/Y$; or (ii) $e$ is a coloop of $M\setminus X/Y$. But, since $X \cup e$ is coindependent in $M$, (ii) cannot occur. Likewise, (i) cannot occur, otherwise $\{b_i, e\}$ is a cocircuit of $M\setminus X/Y$, so $b_i$ is a coloop of $(M\setminus X/Y)\setminus e$ and hence of $N'$; a contradiction. We conclude that (4.7.5) holds.

Next we shall prove that

4.7.6. $M\setminus X/Y$ is 3-connected.

Since $(M\setminus X/Y)\setminus e$ is connected and $e$ is in a triad of $M\setminus X/Y$, the last matroid is certainly connected. Assume that (4.7.6) fails and let $\{Z, W\}$ be a partition of $E((M\setminus X/Y)\setminus e)$ such that $\{Z \cup e, W\}$ is a 2-separation of $M\setminus X/Y$. Then $\{Z, W\}$ is a 2-separation of $(M\setminus X/Y)\setminus e$ unless $|Z| = 1$. But, in the exceptional case, $Z \cup e$ is either (i) a 2-element cocircuit, or (ii) a 2-element circuit of $M\setminus X/Y$. The first case implies the contradiction that $Z$ is a 1-element cocircuit of the connected matroid $(M\setminus X/Y)\setminus e$. In the second case, orthogonality and (4.7.5) imply that $Z$ meets the $h$ disjoint sets of the form $T_i^* - e$. This is a contradiction since $|Z| = 1$ and $h \geq 2$.

We conclude that $\{Z, W\}$ is indeed a 2-separation of $(M\setminus X/Y)\setminus e$.

By (4.7.4), one of $Z$ and $W$ is $\{a_i, b_i\}$ for some $i$. Suppose first that $Z = \{a_i, b_i\}$. Then $Z \cup e = T_i^*$ and, in $M\setminus X/Y$, we have
\[1 = r(Z \cup e) + r(W) - r(M\setminus X/Y) = r(Z \cup e) + r(T_i^* - e) - |Z \cup e| = r(Z \cup e) + 2 - 3.\]

Hence $r(Z \cup e) = 2$. But $r(Z) = r(\{a_i, b_i\}) = 2$ since $a_i$ and $b_i$ are in series in $(M\setminus X/Y)\setminus e$. Thus $e$ is in a circuit $C$ of $M\setminus X/Y$ contained in $T_i^*$. This contradicts orthogonality since, for $j \neq i$, the triad $T_j^*$ of $M\setminus X/Y$ meets $C$ in a single element. We conclude that $Z \neq \{a_i, b_i\}$.

We may now assume that $W = \{a_i, b_i\}$. Since $\{Z \cup e, W\}$ is a 2-separation of $M\setminus X/Y$, it follows that $\{a_i, b_i\}$ is a circuit or a cocircuit of $M\setminus X/Y$. But $\{a_i, b_i\}$ is a proper subset of the triad $T_i^*$ of $M\setminus X/Y$, so $\{a_i, b_i\}$ is certainly not a cocircuit of $M\setminus X/Y$. Moreover, if $\{a_i, b_i\}$ is a circuit of $M\setminus X/Y$, then $b_i$ is a loop of $(M\setminus X/Y)\setminus e/\{a_1, a_2, \ldots, a_h\}$, that is, of $N'$. This contradiction completes the proof of (4.7.6).

By (4.7.2), $r(A \cup Y) = r(A) + r(Y)$, so $(M\setminus X/Y)|A = M|A$. Since $(M, A)$ is a minimal pair, it follows that $X = Y = \emptyset$. Hence $N' = M'$ and so $(M', A')$ is a minimal pair.

5. Proofs of the main theorems

Theorems 1.5 and 1.6 can be combined into the following result.

5.1. Theorem. If $(M, A)$ is a minimal pair, then
\[
|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + \min\{\lambda_3(A, M), r(M) - r_M(A)\} - \alpha(A, M),
\]
where
\[
\alpha(A, M) = \begin{cases} 0, & \text{when } A \text{ is a circuit of } M \\ 1, & \text{when } A \text{ is not a circuit} \end{cases}
\]

This theorem will be derived from two propositions. The first of these, which is stated and proved below, gives the structure of a minimal counterexample to the theorem; the second, which was proved in an earlier paper [6], will be stated following the proof of the first.
$M$ containing $y$ such that $T^*_y - y \subseteq A$. Thus, by orthogonality with the triangle \{x, y, a\}, it follows that $T^*_y = \{a', y, a''\}$ for some $a'' \in A$. Hence either $T_1^*, T, T_2^*$ is a fan of $M$ and (iii) of Lemma 4.3 holds, or $T_1^*, T, T_2^*$ is not a fan of $M$. We may therefore assume that the latter possibility holds. Then, as $|E(N)| \geq 4$ and so $|E(M)| \geq 7$, there is a triangle $T'$ of $M$ such that $T', T_1^*, T, T_2^*$ or $T_1^*, T, T_2^*, T'$ is a chain of $M$. The first case implies that $a \in T'$ and the second that $a'' \in T'$. But $M$ has no triangle containing a so $a'' \in T'$. Hence $T' \cap (T_1^* \cup T \cup T_2^*) = \{y, a''\}$. But $N = M/a, y \backslash x$ and so $N$ has a circuit of size at most two containing $a''$; a contradiction. This completes the proof of Lemma 4.3.

\[\square\]

4.7. Lemma. Suppose that $(M, A)$ is a minimal pair in which $|A| \geq 7$. Let $e$ be an element of $E(M) - cl(A)$ that is in exactly $h$ triads, $T_1^*, T_2^*, \ldots, T_h^*$, for some $h \geq 2$. Suppose that $T^*_i \cap A = \{a_i, b_i\}$ for all $i$. Then $(M', A')$ is a minimal pair where $M' = M/e \backslash \{a_1, a_2, \ldots, a_h\}$ and $A' = A - \{a_1, a_2, \ldots, a_h\}$.

Proof. We note first that, by (vi) of Lemma 4.1, if $i \neq j$, then $T^*_i \cap T^*_j = \{e\}$. Moreover, by (i) of Lemma 4.1, $M'$ is 3-connected. Now $M'$ has a 3-connected minor $N'$ such that $(N', A')$ is a minimal pair and $N'|A' = M'|A'$. Then $N' = M' \backslash X/Y$ for some independent set $Y$ and coindependent set $X$. Clearly

4.7.1. $X \cup e$ is coindependent in $M$.

We shall complete the proof of the lemma by showing that both $X$ and $Y$ are empty. First we show that

4.7.2. $r(A) + r(Y) = r(A \cup Y)$.

By orthogonality, $\{a_1, a_2, \ldots, a_h\}$ is independent in $M/e$ and hence in $M$. As $Y$ is independent in $M/e \backslash \{a_1, a_2, \ldots, a_h\}$, it follows that $\{a_1, a_2, \ldots, a_h\} \cup Y$ is independent in $M$. Thus, as $N'|A' = M'|A'$, we have

$r_{M'/\{a_1, a_2, \ldots, a_h\}}(A - \{a_1, a_2, \ldots, a_h\}) = r_{M'/\{a_1, a_2, \ldots, a_h\}}(A - \{a_1, a_2, \ldots, a_h\})$.

Therefore

$r(A \cup Y) - r(\{a_1, a_2, \ldots, a_h\} \cup Y) = r(A) - r(\{a_1, a_2, \ldots, a_h\}),$

so $r(A \cup Y) - h - |Y| = r(A) - h$, and (4.7.2) follows.

Next we observe that

4.7.3. $|E(N')| \geq 4$.

To see this, we note that, since $|A| \geq 7$ and

$E(N') \supseteq A - \{a_1, a_2, \ldots, a_h\} \supseteq \{b_1, b_2, \ldots, b_h\},$

it follows that $|A| - h \geq h$. Thus, if $h \leq 3$, then $|A| - h \geq 4$, and (4.7.3) holds; and if $h \geq 4$, then $|E(N')| \geq h \geq 4$, and again (4.7.3) holds.

Now $M\backslash(X \cup e)/Y$ is obtained from the 3-connected matroid $N'$ by adding $a_i$ in series with $b_i$ for all $i$. Hence $M\backslash(X \cup e)/Y$ is connected. Moreover, the following is an easy consequence of the fact that $|E(N')| \geq 4$.

4.7.4. Every 2-separation of $M\backslash(X \cup e)/Y$ has some $\{a_i, b_i\}$ as a part.

We show next that

4.7.5. $T^*_i$ is a triad of $M\backslash X/Y$ for every $i$.\]
4.5.4. $Z \cup y$ is a circuit of $M$, the element $n$ exists and is in $A$, and $Z$ equals \{x, a, n\} and is a triad of $M$.

As $Z$ spans $y$ by (7), it follows that $Z \cup y$ is a circuit of $M$, since $y$ does not belong to a triangle and $|Z| = 3$. By Lemma 4.4, there is a triad $T^*_x$ of $M$ such that $x \in T^*_x$ and $T^*_x - x \subseteq A$. By orthogonality, since $x \in Z$, we have that $|T^*_x \cap (Z \cup y)| \geq 2$. As $y \notin A$, it follows that $T^*_x$ is a triad of $H$, which equals $M/y$. Since $Z$ is closed in $H^*$, we deduce that $T^*_x \subseteq Z$ because $1 \geq |T^*_x - (Z \cup y)| = |T^*_x - Z|$. Hence $T^*_x = Z$. Thus $T^*_x = T^*_x \cap A = Z \cap A \subseteq \{a, n\}$, by (6). Hence $n$ exists and is in $A$, and $T^*_x = \{a, n, x\}$. We conclude that (4.5.4) holds.

Now $y \notin \cl(A)$ by (4.3.1). Thus, by (xi) of Lemma 4.1, there is a triad $T^*$ of $M$ such that $y \in T^*$ and $T^* - y \subseteq \cl(A)$. By (4.5.4), $Z$ is also a triad of $M$ and $Z \cup y$ is a circuit of $M$. By (vi) of Lemma 4.1, $|Z \cap T^*| = 1$. If $x \in \cl(A)$, then all but one element of the circuit $Z \cup y$ is in $\cl(A)$; a contradiction. Thus $x \notin \cl(A)$ so, by Lemma 4.1(i), $M/x$ is not 3-connected. Thus we may apply the dual of Lemma 2.7 to $M$ to deduce that $x$ is in some triangle $T$ of $M$. By orthogonality, $|T \cap Z| \geq 2$ and hence $|T - Z| \leq 1$. As $Z$ is closed in $M/y$, it follows that $T \subseteq Z$. Hence $T = Z$ and $Z$ is both a triangle and a triad of the 3-connected matroid $M$. This contradiction implies that Lemma 4.5 holds.

4.6. Lemma. $|Y| \leq 1$ and if $|Y| = 1$, then $|X| = 1$. Moreover, if $Y = \{y\}$ and $X = \{x\}$, then there is an element $a'$ of $A - a$ such that $\{x, a', a\}$ is a triad of $M$.

Proof. Suppose that $y \in Y$. By Lemma 4.5, $M$ has a triangle $T_y$ containing $y$. Now, by (4.3.1), $y \notin \cl(A)$. Thus, by (4.3.7), $T_y$ contains a unique element $a'$ of $A$. Moreover, $a' \neq a$ since, by hypothesis, $a$ is in no triangles. Hence $a' \in E(N)$. Let $T_y = \{y, a', d\}$ where $d$ is some element of $E(M)$. Then $a'$ and $d$ are parallel in $M/y$. But $N$ is a 3-connected minor of $M/y$ and has no circuits of size less than three. Hence $d \notin E(N)$. Now $d \notin Y$, otherwise $a'$ is a loop of $N$. Therefore $d \notin X$. Thus, by Lemma 4.4, $d$ is the unique element $x$ of $X$, and $M$ has a triad $T^*_x$ where $T^*_x = \{x, a, b\}$ for some $b \in A$. Since $x \in T^*_x \cap T_y$, orthogonality implies that $a' = b$ so $a$ is in no triangles. Thus $T_y = \{x, a, b\}$. But $y$ was arbitrarily chosen in $Y$. Hence if $y' \in Y - y$, then $M\{y, y', x, b\} \cong \uu_{2,4}$ and so $\{y, y', x\}$ is a triangle of $M$ meeting the triad $\{x, a, b\}$ in a single element, $x$; a contradiction. We conclude that $Y - y$ is empty. Hence $|Y| \leq 1$. Moreover, if $|Y| = 1$, say $Y = \{y\}$, then $X = \{x\}$ and the required element, $x$, and triad and triad of $M$ exist.

We are now ready to finish the proof of Lemma 4.3. We have $N = (M/a) \setminus X/Y$. By Lemma 4.4, $|X| \leq 1$, and, by Lemma 4.6, $|Y| \leq 1$ with equality in the latter implying equality in the former. Hence we have three possibilities:

(i) $|Y| = 0 = |X|$;
(ii) $|Y| = 0$ and $|X| = 1$; and
(iii) $|Y| = 1 = |X|$.

The first possibility means that (i) of Lemma 4.3 holds. Moreover, by Lemma 4.4, the second possibility implies that (ii) of Lemma 4.3 holds. Thus it remains only to consider when the third possibility holds. Hence let $Y = \{y\}$ and $X = \{x\}$. By Lemma 4.6, for some $a'$ in $A - a$, the matroid $M$ has a triad $T^*_x$ and a triangle $T$ such that $T^*_x = \{a, x, a'\}$ and $T = \{x, y, a'\}$. Since $y \notin \cl(A)$, the triangle $T$ implies that $x \notin \cl(A)$. Moreover, by (xi) of Lemma 4.1 and (4.3.6), there is a triad $T^*_z$ of
Recall that \( N = (M/a) \setminus X / Y \). We have just shown that \( |X| \leq 1 \). Next we consider \( Y \).

4.5. Lemma. If \( y \in Y \), then \( y \) belongs to a triangle of \( M \).

Proof. Suppose that \( y \) is in no triangles of \( M \). By (4.3.1), \( y \not\in \text{cl}(A) \). Thus, if \( H = M/y \), then \( H \) has a 2-separation \( \{Z, W\} \). By (4.3.5), we may assume that \( \{Z, W\} \) is chosen such that \( |Z \cap E(N)| \leq 1 \) and \( Z \) is closed in both \( H \) and \( H^* \). As before, if \( Z \cap E(N) \) is non-empty, its unique element will be denoted by \( n \). Thus
\[
Z \cap A \subseteq \{a, n\}. \tag{6}
\]
As \( M \) is 3-connected and \( \{Z, W\} \) is a 2-separation for \( M/y \), it follows that
\[
y \in \text{cl}(Z) \cap \text{cl}(W) \tag{7}
\]
We shall prove next that

4.5.1. \( Z - X \) is independent in \( M/y \setminus X \).

We argue by contradiction. Thus assume that \( M/y \) has a circuit \( C \) that is contained in \( Z - X \). By (4.3.2), it follows that \( C - (Y \cup a) \neq \emptyset \). Hence
\[
\emptyset \neq C - (Y \cup a) \subseteq Z - (X \cup Y \cup a) \subseteq Z \cap E(N) \subseteq \{n\}.
\]
Thus \( n \) exists and is in \( C \), and \( C - (Y \cup a) = \{n\} \). Hence \( n \) is a loop of \( (M/y)/(C-n) \) and \( y \cup (C-n) \subseteq Y \cup a \) so \( N \) is a minor of \( (M/y)/(C-n) \). Therefore \( n \) is a loop of \( N \); a contradiction. We conclude that (4.5.1) holds.

Now, we shall prove that

4.5.2. \(|Z| \geq 3 \) and \( X \) contains a unique element \( x \). Moreover, \( x \in Z \) and, for every 2-element subset \( Z_0 \) of \( Z - X \), the set \( Z_0 \cup x \) is a triad of \( M \).

The fact that \(|Z| \geq 3 \) follows because \( Z \) spans \( y \) in \( M \) by (7), and \( y \) does not belong to a triangle. Now, by Lemma 4.4, \(|X| \leq 1 \). Moreover, since \(|Z \cap E(N)| \leq 1 \) and \(|E(N)| \geq 4 \), we have
\[
|W| \geq |E(N) - Z| \geq |E(N)| - 1 \geq 3.
\]
Thus \(|W - X| \geq 2 \) and, since \(|Z| \geq 3 \), we also have that \(|Z - X| \geq 2 \). Hence \( \{Z - X, W - X\} \) is a 2-separation for \( M/y \setminus X \). Therefore, by (4.5.1) and Lemma 2.5, if \( Z_0 \) is a 2-element subset of \( Z - X \), then \( Z_0 \) contains a cocircuit of \( M/y \setminus X \). Thus \( Z_0 \cup X \) contains a cocircuit of \( M/y \) and hence of \( M \). But every cocircuit of \( M \) has at least three elements, so \( Z_0 \cup X \) is a cocircuit of \( M \) and \(|X| = 1 \) since, by Lemma 4.4, \(|X| \leq 1 \). Moreover, \( X \subseteq Z \) since \( Z \) is closed in \( H^* \).

Now, we shall show that

4.5.3. \(|Z| = 3 \).

Suppose that \(|Z| \geq 4 \). By (4.5.2), there is a unique element \( x \) in \( X \), and \( M^*|Z \) is isomorphic to \( U_{2,1,|Z|}^1 \). If \( x \not\in \text{cl}(A) \), it follows that \( M \) has two triads that contradict (vi) of Lemma 4.1. Thus \( x \in \text{cl}(A) \). Now let \( Z_0 \) be a 2-element subset of \( Z - x \). By (4.5.2), \( x \cup Z_0 \) is a triad of \( M \). Since \( x \in \text{cl}(A) - A \), it follows by Lemma 2.6 that \(|(x \cup Z_0) \cap \text{cl}(A)| \geq 2 \). Thus, by hypothesis (b), \(|(x \cup Z_0) \cap A| \geq 2 \). As \( x \not\in A \), it follows that \( Z_0 \subseteq A \). As \( Z_0 \) was chosen arbitrarily, we deduce that \( Z - x \subseteq A \). Thus \( Z - x = Z \cap A \). But, by (6), \( Z \cap A \subseteq \{a, n\} \). Thus \( 3 \leq |Z - x| = |Z \cap A| \leq 2 \); a contradiction. Hence (4.5.3) holds.

Next we prove the following.
A spans an element \( z \) of \( Z - \{a, n\} \). Then \( \{a, n, z\} \subseteq \text{cl}(A) \) and \( A \) spans \( Z \) because \( \{a, n, z\} \) is a basis for \( Z \), since, by hypothesis, \( a \) does not belong to a triangle of \( M \). As \( \text{cl}(A) \cap Y = \emptyset \) by (4.3.1), \( Z \cap Y = \emptyset \). But \( Z - \{a, n\} \) avoids \( E(N) \) so \( Z - \{a, n\} \subseteq X \). Thus \( N \) is a minor of \( M \backslash \{Z - \{a, n\}\} \), which equals \( M \backslash \{W \cup \{a, n\}\} \). Since \( n \) is a coloop in the last matroid by (4.4.3), \( n \) is a coloop of \( N \); a contradiction. Hence \( A \) cannot span any element of \( Z - A \) and so \( (Z - A) \cap \text{cl}(A) = \emptyset \).

4.4.5. \( r(Z) = |Z| \).

Suppose that \( |Z| \geq r(Z) + 1 \). Then \( M[Z] \) has a circuit, \( C \). Observe that \( C - A = C - \{a, n\} \) since, by (4.4.4), \( n \) exists and belongs to \( A \) and by (4.4.1), \( Z \cap A \subseteq \{a, n\} \). Now \( |C - \{a, n\}| \geq 2 \), otherwise \( \{a, n\} \subseteq C \) and \( |C| = 3 \) which is contrary to the fact that \( a \) does not belong to a triangle. Let \( z_1 \) and \( z_2 \) be distinct elements of \( C - A \), and suppose that \( z \in \{z_1, z_2\} \). Then \( z \in Z - A \), so, by (4.4.4), \( z \not\in \text{cl}(A) \). Moreover, by (xi) of Lemma 4.1, \( M \) has a triad \( T^* \) that contains \( z \) and meets \( \text{cl}(A) \) in two elements. By (4.3.6), \( T^* \) must meet \( A \) in two elements. By orthogonality, \( |T^* \cap C| \geq 2 \). Hence \( |T^* - C| \leq 1 \) and so \( T^* \subseteq Z \) by (4.3.5). Thus \( T^* \cap A \subseteq Z \cap A \subseteq \{a, n\} \) by (4.4.1). Hence \( T^* = \{z, a, n\} \). In particular, the triads \( \{z_1, a, n\} \) and \( \{z_2, a, n\} \) contradict (vi) of Lemma 4.1.

Next we complete the proof of the first part of Lemma 4.4 by showing that \( T^* \) exists, thereby obtaining a contradiction since we have assumed that \( T^* \) does not exist.

4.4.6. \( T^* \) exists.

Suppose first that \( x \in X - \text{cl}(A) \). Then, by hypothesis (a), \( M \) has no triangle that contains \( x \) and avoids \( \text{cl}(A) \). Therefore, by (xi) of Lemma 4.1, there is a triad \( T^* \) of \( M \) such that \( T^* - x \subseteq \text{cl}(A) \) and hence, by (4.3.6), \( T^* - x \subseteq A \). If \( a \in T^* \), we can take \( T^* = T^*_x \) and (4.4.6) holds. Thus we may assume that \( a \not\in T^* \). Then \( T^* - x \) is a 2-cocircuit of \( M \backslash x/a \). As \( N \) is a minor of \( M \backslash x/a \) and \( T^* - x \subseteq \text{cl}(A) - a \), the cocircuit \( T^* - x \) contradicts (4.3.4). We conclude that, for every element \( x \) of \( X \) \( \text{cl}(A) \), the triad \( T^*_x \) exists.

Now suppose that \( x \in \text{cl}(A) - A \). By (4.4.5), \( r(Z) = |Z| \). As \( \{Z, W\} \) is a 2-separation for \( M \backslash x \), we have, by Lemma 2.5, that if \( z \) and \( z' \) are distinct elements of \( Z \), then \( \{x, z, z'\} \) is a cocircuit of \( M \). But \( x \in \{x, z, z'\} \cap (\text{cl}(A) - A) \), so, by Lemma 2.6, \( |\{x, z, z'\} \cap \text{cl}(A)| \geq 2 \). Hence, by hypothesis (b), \( |\{x, z, z'\} \cap A| \geq 2 \). As \( x \not\in A \), it follows that both \( z \) and \( z' \) belong to \( A \). Since these two elements were arbitrarily chosen in \( Z \), it follows that \( Z \subseteq A \). Thus \( Z \cap A = Z \). By (4.4.1), \( Z \cap A \subseteq \{a, n\} \) and hence \( Z = \{a, n\} \) since \( |Z| \geq 2 \). In this case, we can take \( T^*_x = \{x, a, n\} \) thereby completing the proof of (4.4.6) and hence that of Lemma 4.4(i).

To prove the second part of the lemma, we shall argue by contradiction. Suppose that \( x \) and \( x' \) are different elements of \( X \). By (i), the triads \( T^*_x \) and \( T^*_y \) both exist. Let \( T^*_x = \{a, x, b\} \) and \( T^*_y = \{a, x', b'\} \) be as in Lemma 4.4. Thus \( \{b, b'\} \subseteq A \).

We prove next that \( b \neq b' \). Assume the contrary. Then, by (vi) of Lemma 4.1, both \( x \) and \( x' \) belong to \( \text{cl}(A) \). Moreover, \( M' \{a, x, x', b\} \) is isomorphic to \( U_{2, 4} \). Hence \( \{x, x', b\} \) is a triad of \( M \) containing two elements of \( \text{cl}(A) \) but just one element of \( A \); a contradiction to hypothesis (b). Thus \( b \neq b' \).

Observe that \( \{b, b', a\} \) is contained in a series class or is a set of coloops of \( M \backslash \{x, x'\} \). Thus \( \{b, b'\} \) contains a cocircuit of \( M \backslash a \backslash \{x, x'\} \) which is contrary to (4.3.4).
4.4.1. \( Z \cap A \subseteq \{a, n\} \), where \( n \) may not exist.

Next we show that

4.4.2. \( W \cup A \) is a spanning set of \( M \).

Suppose that \( W \cup A \) does not span \( M \). Then there is a cocircuit \( D \) of \( M \) that avoids \( \text{cl}(W \cup A) \). Since \( D \) must also avoid \( \text{cl}(A) \), Lemma 4.1(ii) implies that every element of \( D \) is essential. Thus, by Theorem 2.8, \( M \) has two distinct triangles \( T_i \) and \( T_j \) meeting \( D \). By (4.3.7), each \( T_i \) contains a unique element \( a_{T_i} \) of \( \text{cl}(A) \) and \( a_{T_i} \in A \). Moreover, by Lemma 4.1(ix), \( a_{T_i} \neq a_{T_j} \). But, for each \( i \), we have \( T_i - a_{T_i} \subseteq D \). Moreover, \( D \subseteq Z \) since \( D \) avoids \( \text{cl}(W) \). Since, by (4.3.5), \( Z \) is closed in \( M \setminus x \), it follows that \( a_{T_i} \in Z \) for each \( i \). Hence, by (4.4.1), \( \{a_{T_1}, a_{T_2}\} \subseteq Z \cap A \subseteq \{a, n\} \). Therefore \( a = a_{T_j} \) for some \( j \), so \( a \) is in a triangle; a contradiction to the hypothesis.

4.4.3. \( 2 \leq r(Z) \leq 3 \). Moreover, if \( r(Z) = 3 \), then \( n \) exists and is in \( A \), and both \( a \) and \( n \) belong to \( Z \) and are coloops of \( M \setminus W \cup \{a, n\} \); and if \( a \in W \), then \( r(Z) = 2 \), and \( n \) exists, is in \( A \), and is a coloop of \( M \setminus W \cup n \).

By (4.4.1), \( W \cup A \subseteq W \cup \{a, n\} \) where \( n \) may not exist. As \( \{Z, W\} \) is a 2-separation of \( M \setminus x \), we have \( \min \{|Z|, |W|\} \geq 2 \) and \( r(Z) + r(W) = r(M) + 1 \). But \( M \setminus x \) is simple, so

\[
2 \leq r(Z) = r(M) - r(W) + 1. \tag{3}
\]

Moreover, by (4.4.2),

\[
r(M) = r(W \cup A) = r(W \cup (A - W)) \leq r(W) + |A - W|.
\]

Hence

\[
r(W \cup A) - r(W) = r(M) - r(W) \leq |A - W|. \tag{4}
\]

Substituting for \( r(M) - r(W) \) into (3), we get

\[
2 \leq r(Z) \leq |A - W| + 1. \tag{5}
\]

Since \( A - W \subseteq \{a, n\} \), where \( n \) may not exist, we deduce that \( 2 \leq r(Z) \leq 3 \). Moreover, if \( r(Z) = 3 \), then \( |A - W| = 2 \) and equality holds in (4). Hence \( n \) exists, both \( a \) and \( n \) are in \( Z \), and both are coloops of \( M \setminus W \cup \{a, n\} \). On the other hand, if \( a \in W \), then \( |A - W| = 1 \), so \( n \) exists and equality holds throughout both (5) and (4). Thus \( r(Z) = 2 \), and \( n \) is a coloop of \( M \setminus W \cup n \).

4.4.4. If \( |Z| \geq r(Z) + 1 \), then \( (Z - A) \cap \text{cl}(A) \) is empty, and \( n \) exists and is in \( A \).

Suppose first that \( r(Z) = 2 \). Then \( M \setminus Z \) is isomorphic to \( U_{2,|Z|} \). As \( a \) does not belong to a triangle, it follows that \( a \in W \). Hence, by (4.4.3), \( n \) exists and is in \( A \). Thus, by (4.4.1), \( Z - A = Z - n \). Now suppose that \( A \) spans an element \( z \) of \( Z - n \). Then \( \{n, z\} \subseteq \text{cl}(A) \) and \( A \) spans \( Z \) because \( \{n, z\} \) is a basis for \( Z \). As \( \text{cl}(A) \cap Y = \emptyset \) by (4.3.1), \( Z \cap Y = \emptyset \). But \( Z - n \) avoids \( E(N) \). Hence \( Z - n \subseteq X \). Thus \( N \) is a minor of \( M \setminus (Z - n) \) which equals \( M \setminus W \cup n \). But \( n \) is a coloop of the last matroid by (4.4.3), so \( n \) is a coloop of the 3-connected matroid \( N \); a contradiction. We conclude that \( A \) cannot span any element of \( Z - A \), and so \( (Z - A) \cap \text{cl}(A) = \emptyset \) when \( r(Z) = 2 \).

We may now assume that \( r(Z) \geq 3 \). Then, by (4.4.3), \( r(Z) = 3 \), \( n \) exists, and \( a \) and \( n \) both belong to \( Z \cap A \). Thus, by (4.4.1), \( Z - A = Z - \{a, n\} \). Suppose that
The second result that we need about a minor $H$ of $M$ having $N$ as a minor is the following.

4.3.5. If $H$ is not 3-connected, then there is a $k$-separation $\{Z, W\}$ of $H$ with $k$ in $\{1, 2\}$ such that $|Z \cap E(N)| \leq 1$ and $Z$ is closed in both $H$ and $H^*$. 

As $H$ is not 3-connected, it has a $k$-separation $\{Z, W\}$ for some $k$ in $\{1, 2\}$. But $N$ is a 3-connected matroid and is a minor of $H$, so

$$\min\{|Z \cap E(N)|, |W \cap E(N)|\} \leq 1.$$ 

In particular, we may assume that

$$|Z \cap E(N)| \leq 1. \quad (1)$$

Now choose such a $k$-separation satisfying (1) so that $|Z|$ is as large as possible. Note that, since $|E(N)| \geq 4$, we must have that

$$|W \cap E(N)| \geq 3. \quad (2)$$

If $Z$ is closed in both $H$ and $H^*$, then (4.3.5) follows. Thus, we may assume that there is an element $w$ of $W$ such that $w$ is spanned by $Z$ in $H$ or $H^*$. Hence $r_H(Z) = r_H(Z \cup w)$ or $r_{H^*}(Z) = r_{H^*}(Z \cup w)$, and so

$$r_H(Z) + r_{H^*}(Z) - |Z| \geq r_H(Z \cup w) + r_{H^*}(Z \cup w) - |Z \cup w|.$$ 

Thus $\{Z \cup w, W - w\}$ is a $j$-separation of $H$ for some $j$ in $\{1, 2\}$. Again, since $N$ is 3-connected, we have $\min\{|(Z \cup w) \cap E(N)|, |(W - w) \cap E(N)|\} \leq 1$. It follows by (2) that $|(Z \cup w) \cap E(N)| \leq 1$. Therefore $\{Z \cup w, W - w\}$ contradicts the choice of $\{Z, W\}$ and we conclude that (4.3.5) holds.

4.3.6. If $T^*$ is a triad of $M$ such that $e \in T^* - \text{cl}(A)$ and $T^* - e \subseteq \text{cl}(A)$, then $T^* - e \subseteq A$.

To see this, note that $T^* \cap \text{cl}(A) = T^* - e$. Hence $|T^* \cap \text{cl}(A)| = 2$ and (4.3.6) follows by hypothesis (b).

4.3.7. If $T$ is a triangle of $M$ such that $T - \text{cl}(A)$ is non-empty, then $T \cap \text{cl}(A)$ contains exactly one element, $a_T$, and $a_T \in A$.

As $T - \text{cl}(A)$ is non-empty but $M \setminus \text{cl}(A)$ has no triangles, it follows that $1 \leq |T - \text{cl}(A)| \leq 2$. But $\text{cl}(A)$ does not span $T$, so $|T - \text{cl}(A)| = 2$. Let $e$ be an element of $T - \text{cl}(A)$. By (xi) of Lemma 4.1, there is a triad $T^*$ of $M$ such that $e \in T^*$ and $T^* - e \subseteq \text{cl}(A)$. By (4.3.6), $T^* - e \subseteq A$. By orthogonality, $T^* \cap T \neq \{e\}$. Thus the unique element $a_T$ of $T \cap (T^* - e)$ must belong to $A$, and (4.3.7) holds.

Recall that $N = (M/A) \setminus X/Y$. To establish Lemma 4.3, we shall show that both $X$ and $Y$ must be small. We begin by considering $X$.

4.4. Lemma. (i) If $x \in X$, then there is a triad $T_x^*$ of $M$ such that $T_x^* = \{x, a, b\}$ for some $b \in A$.

(ii) $|X| \leq 1$.

Proof. The proof of Lemma 4.4(i) is long and will be divided into a number of steps. We shall argue by contradiction. Thus suppose that (i) fails, that is, $T_x^*$ does not exist for some $x \in X$. Let $H = M \setminus x$ and observe that $H$ is not 3-connected since $(M, A)$ is a minimal pair. Therefore $H$ has a 2-separation $\{Z, W\}$ satisfying (4.3.5). Hence $Z \cap E(N)$ has at most one element. Moreover, when this element exists, we shall denote it by $n$. It is not difficult to see that
4.3. Lemma. Suppose that \((M, A)\) is a minimal pair in which \(A\) is non-spanning such that
(a) \(M \setminus \mathrm{cl}(A)\) has no triangles; and
(b) every triad with at least two elements in \(\mathrm{cl}(A)\) contains at least two elements of \(A\).

Let \(a\) be an element of \(A\) that is in no triangles of \(M\), and \(N\) be a minimal 3-connected minor of \(M/a\) for which \(N|(A - a) = (M/a)|(A - a)\). Then
(i) \(N = M/a\); or
(ii) \(N = M/a\setminus e\) for some element \(e\) of \(E(M) - A\) such that \(\{a, e\}\) is contained in a triad of \(M\) whose third element is in \(A\); or
(iii) \(N = M/\{a, e\}\setminus e'\) for some elements \(e\) and \(e'\) of \(E(M) - \mathrm{cl}(A)\) such that \(\{e, a_1, e'\}\) is a triangle of \(M\) and \(\{a, e, a_1\}, \{e, a_1, e'\}, \{a_1, e', a_2\}\) is a fan of \(M\) for some elements \(a_1\) and \(a_2\) of \(A\).

Proof. Suppose that \(N = (M/a)\setminus X/Y\) for some disjoint subsets \(X\) and \(Y\) of \(E(M) - a\). We show first that

4.3.1. \(Y \cap \mathrm{cl}(A) = \emptyset\).

Assume that (4.3.1) fails and let \(y\) be an element of \(Y \cap \mathrm{cl}(A)\). As \(N|(A - a) = (M/a)|(A - a)\) and \(N\) is a minor of \(M/\{a, y\}\), it follows that
\[ r_{M/a}(A - a) = r_{N}(A - a) \leq r_{M/\{a, y\}}(A - a) = r_{M/a}((A - a) \cup y) - r_{M/a}(y). \]

Thus \(y \not\in \mathrm{cl}_{M/a}(A - a)\) so \(y \not\in \mathrm{cl}(M)\); a contradiction. Hence (4.3.1) holds.

Now choose \(X\) and \(Y\) such that \(|X|\) is as large as possible. Note that

4.3.2. \(Y \cup a\) is independent in \(M\).

If not, there is a circuit \(C\) of \(M\) such that \(C \subseteq Y \cup a\). Take \(y \in C - a\). Then \(y\) is a loop of \(M/[(Y - y) \cup a]\). Hence \(N = (M/a)\setminus (X \cup y)/(Y - y)\), which is contrary to the choice of \(X\) and \(Y\).

We show next that we may also suppose that

4.3.3. \(|E(N)| \geq 4\).

Suppose not. Then \(|E(N)| \leq 3\) and so, as \(N\) is 3-connected, \(r(N) \leq 2\). Thus
\[ r_{M}(A) = r_{M/a}(A - a) + 1 = r_{N}(A - a) + 1 \leq r(N) + 1 \leq 3 \]
where the second equality holds since \(N|(A - a) = (M/a)|(A - a)\). But now, by (x) of Lemma 4.1, we conclude that \(M\) is isomorphic to a rank-four wheel having \(\mathrm{cl}(A)\) as its rim. Hence \(a\) is in a triangle of \(M\); a contradiction.

Next we prove the following,

4.3.4. Let \(H = M\setminus X_1/Y_1\) where \(X_1 \subseteq X\) and \(Y_1 \subseteq Y \cup a\). Then \(H\) cannot have a cocircuit \(D\) such that \(|D| \leq 2\) and \(D \subseteq \mathrm{cl}(A) - a\).

Suppose that such a cocircuit \(D\) does exist. We shall first prove that \(D \cap A \neq \emptyset\). Assume that \(D \cap A = \emptyset\) and let \(d\) be an element of \(D\). As \(d \in \mathrm{cl}(A) - A\), there is a circuit \(C\) of \(M\) such that \(d \in C \subseteq A \cup d\). Observe that, since \(A - a \subseteq E(H)\) and \(a \not\in X_1\), the set \(C\) contains a circuit \(C'\) of \(H\) such that \(d \in C' \subseteq A \cup d\). Hence \(C' \cap D = \{d\}\), which is contrary to orthogonality. Thus \(D \cap A \neq \emptyset\). Let \(a'\) be an element of \(D \cap A\). Observe that \(a'\) must belong to a cocircuit \(D'\) of \(N\) such that \(D' \subseteq D\). Thus \(|D'| \leq |D| \leq 2\), which contradicts (4.3.3) since \(N\) is 3-connected. Hence (4.3.4) holds.
Hence, by Tutte’s triangle lemma (2.4), $M$ has a triangle $T$ that meets $T^*$ in exactly two elements, one of which is $e$. Let $T = \{e, f, g\}$. We may assume that $f \not\in \text{cl}(A)$. Therefore, by hypothesis, $g \in \text{cl}(A)$. By (vii), $M$ has a triad $T^*_e$ that meets $\{e, f\}$ in $\{e\}$. Orthogonality implies that $g \in T^*_e$, and (iv) implies that $T^*_e - T \subseteq \text{cl}(A)$. Thus $|T^*_e \cap \text{cl}(A)| = 2$, as desired.

4.2. Lemma. Let $(M, A)$ be a minimal pair and $T$ be a triangle of $M$ that avoids $\text{cl}(A)$. If $a$ is a member of $A$ that belongs to a triad meeting $T$, then $(M/T, A - a)$ is a minimal pair.

Proof. Let $T = \{e_1, e_2, e_3\}$. Then, by (viii) of Lemma 4.1, for each $i$ in $\{1, 2, 3\}$, there is a unique triad $T^*_i$ of $M$ such that $T^*_i = (T - e_i) \cup a_i$ for some $a_i$ in $A$. Thus $a = a_i$ for some $i$ and, by symmetry, we may assume that $i = 3$. Let $M' = M/T$ and $A' = A - a_3$. By Lemma 4.1(viii)(b), $M'$ is 3-connected. We may suppose that $(M', A')$ is not a minimal pair. Then there is a proper 3-connected minor $N'$ of $M'$ such that $N'|A' = M'|A'$ and $(N', A')$ is a minimal pair. Clearly $N' = M\setminus X/Y$ for some disjoint subsets $X$ and $Y$ of $E(M) - (A' \cup T)$ such that $X \cup Y$ is non-empty. Observe that $a_3 \not\in Y$ because Lemma 4.1(viii)(b) implies that $a_3$ is spanned by $A'$ in $M'$.

We may now assume that $a_3 \in X \cup E(N')$. Then, since $\{a_1, a_2, a_3\}$ is a triad of $M'$, either

(a) $a_3 \in E(N')$ and $\{a_1, a_2, a_3\}$ is a triad of $N'$; or

(b) $\{a_1, a_2, a_3\}$ properly contains a cocircuit of $N'$.

In case (b), since $N'$ is 3-connected and $\{a_1, a_2\}$ has rank two in it, we deduce that $N'$ must be a triangle. As $a_1$ and $a_2$ are coloops of $M'|A'$ and $M'|A' = N'|A'$, it follows that $A' = \{a_1, a_2\}$ and $|A| = 3$. This cannot happen by (x) of Lemma 4.1 since the rank-four wheel has no triangle avoiding the closure of its rim. Hence (a) must hold. Now, by Lemma 4.1(viii)(b), $M^* = P_\Delta(M(K_4), M^*\setminus T)$ where $\Delta = \{a_1, a_2, a_3\}$ and $M(K_4)$ is labelled as in Figure 1; and $(N')^* = (M^*\setminus T)/X\setminus Y$. Since $\Delta$ is a triangle of $(N')^*$, it follows that

$$M^*/X\setminus Y = P_\Delta(M(K_4), M^*\setminus T/X\setminus Y) = P_\Delta(M(K_4), (N')^*).$$

Let $N^* = M^*/X\setminus Y$. Since $(N')^*$ and $M(K_4)$ are 3-connected having at least three common elements, by [10], $N^*$ is 3-connected. Hence $N$ is 3-connected.

We shall show next that

4.2.1. $N|A' = M|A'$.

To see this, first note that $M'|A' = M|A'$ by Lemma 4.1(viii)(c). Moreover, by assumption, $N'|A' = M'|A'$. Thus

$$r_M(A') = r_{M'}(A') = r_{N'}(A').$$

But, since $N'$ is a minor of $N$, and $N$ is a minor of $M$, we have

$$r_M(A') \geq r_N(A') \geq r_{N'}(A').$$

Therefore equality holds throughout the last line, so $r_N(A') = r_M(A')$. As $N$ is a minor of $M$, it is now easy to check that $N|A' = M|A'$. Hence (4.2.1) holds.

Since $a_3$ is a coloop of both $N|A$ and $M|A$, it follows from (4.2.1) that $N|A = M|A$. Therefore, as $(M, A)$ is a minimal pair, $N = M$ and so $X \cup Y$ is empty and $N' = M'$. Thus $(M', A')$ is a minimal pair. $\square$.
Suppose that \( T' \) has a member \( T \) such that \( T \cap \text{cl}(A) = \emptyset \). Let \( T = \{ e_1, e_2, e_3 \} \).

By (viii), for each \( i \in \{1, 2, 3\} \), there is a triad \( T_i^* \) of \( M \) and an element \( a_i \) of \( A \) such that \( T_i^* = (T - e_i) \cup a_i \). Thus \( a_i \) is a coloop of \( M \mid A \) and hence, \( |A| = 3 \) because \( r(A) \leq 3 \) by hypothesis. Therefore \( A = \{a_1, a_2, a_3\} \). If \( f \in \text{cl}(A) - A \), then there is a circuit \( C \) of \( M \) such that \( f \in C \subseteq A \cup f \). Thus \( a_i \in C \) for some \( i \), so \( C \cap T_i^* = \{a_i\} \), which is contrary to orthogonality. Hence \( \text{cl}(A) = A \). Let \( T' \) be in \( T' - \{T\} \). Next we shall prove that \( T' \cap A \neq \emptyset \). Assume the contrary. Then \( T' \) avoids \( T \) otherwise \( |T' \cap T_i^*| = 1 \) for some \( i \). Now, applying (viii)(c) to the triangle \( T' \), it follows that \( A \) is contained in a series class of \( (M/T')\mid A \) and hence, as \( |A| = 3 \), we deduce that \( A \) is a circuit of \( M/T' \). But \( T_i^* \) is a triad of \( M/T' \), a contradiction since \( |A \cap T_i^*| = 1 \). We conclude that \( T' \) does indeed meet \( A \). Thus we may assume that \( a_i \in T' \). By orthogonality, \( e_2 \) or \( e_3 \), say \( e_2 \), belongs to \( T' \). By (vi), \( T \cap T_i^* = \{e_2\} \). As \( T' \cap T_i^* = \{e_2\} \), it follows that \( a_3 \in T' \) and \( T' = \{a_1, e_2, a_3\} \).

This is a contradiction because it implies that \( e_2 \in \text{cl}(A) \), yet \( \text{cl}(A) = A \).

We may now assume that every \( T \) in \( T' \) meets \( \text{cl}(A) \). Thus, in the notation of (ix), \( T' = T \). Let \( T \in T \). Then \( A_T \cup T \) is the union of two triads of \( M \). Thus, in \( M \mid \text{cl}(A) \), either (a) \( A_T \) is contained in a series class, or (b) \( A_T \) is a set of coloops. But, in the latter case, \( A_T \) is an independent set of size three. Since \( A_T \subseteq \text{cl}(A) \), we deduce that \( A_T \) spans \( \text{cl}(A) \). Thus \( A_T = \text{cl}(A) = A \). But \( |T'| \geq 3 \), so there is a \( T' \) in \( T - T \). As \( A_T \) is a 3-element subset of the 3-element set \( \text{cl}(A) \), we have \( A_T = A = A_T \); a contradiction to (ix)(a). Hence we may assume that (a) holds. Then, as \( |A_T| = 3 \), either \( A_T \) is a basis of \( A \), or \( A_T \) is a triangle. In the second case, letting \( X = A_T \cup T \), we have \( r(X) + r^*(X) - |X| \leq 1 \). This contradicts the fact that \( M \) is 3-connected since \( |X| = 5 \) and, by (iii), \( |E(M)| \geq 7 \). We conclude that \( A_T \) is a basis of \( A \) and it is not difficult to see that \( \text{cl}(A) \) is a 4-circuit of \( M \).

Now let \( T \) and \( T'' \) be distinct triangles in \( T \) and let \( Z = \text{cl}(A) \cup T'' \cup T'' \). Since each of \( \text{cl}(A) \), \( T \), and \( T'' \) is a circuit and \( \text{cl}(A) - (T \cup T'') \) is non-empty, we deduce that \( r(Z) \leq |Z| - 3 \). Moreover, each element of \( T \cup T'' \) is in a triad with two elements of \( \text{cl}(A) \). Thus \( r^*(Z) \leq |\text{cl}(A)| = 4 \). Hence

\[
r(Z) + r^*(Z) - |Z| \leq (|Z| - 3) + 4 - |Z| = 1.
\]

Since \( |Z, E(M) - Z| \) is not a 2-separation of \( M \), it follows that \( |E(M) - Z| \leq 1 \) and so \( |E(M)| \leq 9 \). But \( A_T \) and \( A_T'' \) are distinct 3-element subsets of the 4-element set \( \text{cl}(A) \), so \( \text{cl}(A) = A_T \cup A_T'' \). Hence every element of \( \text{cl}(A) \) is in a triad of \( M \). But, by (ii), every element of \( E(M) - \text{cl}(A) \) is also in a triad of \( M \). Hence \( M \) is minimally 3-connected. Moreover, \( r(M) \geq 4 \), so \( |E(M) - \text{cl}(A)| \geq 4 \), otherwise \( E(M) - \text{cl}(A) \) is a triad avoiding \( A \) contradicting (v). Thus \( |E(M)| \geq 8 \).

Suppose that \( r(M) = 4 \). Then, by Oxley [8], since \( M \) is minimally 3-connected, \( |E(M)| \leq 8 \) with equality only if \( M \) is a wheel or a whirl. Therefore \( |E(M)| = 8 \). In a rank-four wheel or a rank-four whirl, the only 4-circuit that is a flat of the matroid is the rim of the rank-four wheel. Thus the result holds when \( r(M) = 4 \).

We may now assume that \( r(M) \geq 5 \). Since every cocircuit avoiding the 4-circuit \( \text{cl}(A) \) has size at least four and \( |E(M)| \leq 9 \), it follows that \( r(M) = 5 \) and \( |E(M)| = 9 \). Thus \( E(M) - \text{cl}(A) \) contains at least two cocircuits of \( M \). By elimination, it follows that every 4-element subset of \( E(M) - \text{cl}(A) \) is a cocircuit. But \( T \) meets \( E(M) - \text{cl}(A) \) in exactly two elements, so \( E(M) - \text{cl}(A) \) contains a cocircuit meeting \( T \) in a single element; a contradiction.

(x) From (ii), there is a triad \( T^* \) of \( M \) such that \( e \in T^* \). Moreover, we may assume that \( |T^* \cap \text{cl}(A)| \leq 1 \), otherwise the result holds. Thus \( |T^* - \text{cl}(A)| \geq 2 \).
that $e_3 \in C' \subseteq (C \cup T) - e_2$. As $C' \cap T^*_T \subseteq \{e_1\}$, it follows that $e_1 \not\in C'$. Hence $e_3 \in C' \subseteq AU e_3$. This is a contradiction because $A$ does not span $e_3$. Thus $Z \subseteq C$. We conclude that (4.1.1).

By (viii)(b), $M/T$, which is isomorphic to $M\setminus e_3/e_1, e_2$, is 3-connected. Since $(M, A)$ is a minimal pair, $M|A \neq (M\setminus e_3/e_1, e_2)|A$. Thus $M$ does indeed have a circuit contained in $A \cup \{e_1, e_2\}$ and meeting $\{e_1, e_2\}$. By (4.1.1), it follows that $Z$ is contained in a series class of $M|A \cup \{e_1, e_2\}$, hence $\{a_1, a_2, a_3\}$ is contained in a series class of $[M|A \cup \{e_1, e_2\}]_{\{e_1, e_2\}}$, which equals $(M/T)|A$. Finally, for each $i$ in $\{1, 2, 3\}$, each member of $Z - a_i$ is a coloop of $[M|A \cup \{e_1, e_2\}]_{\{e_1, e_2\}}$ so $M|(A - a_i) = [M|A \cup \{e_1, e_2\}]_{\{e_1, e_2\}}/[e_1, e_2]\{A - a_i\} = (M/T)|(A - a_i)$.

(ix) If $T \in T$, then, as $T$ meets both $cl(A)$ and $E(M) - cl(A)$, it follows by (vii) that there are triads $T_i^*$ and $T_j^*$ of $M$ such that $T_i^* - T, T_j^* - T$ is a chain of $M$ having both its spokes in $E(M) - cl(A)$. By (iv), it follows that the set $A_T$ of rim elements of this chain is contained in $cl(A)$. Observe that $A_T$ is contained in a series class or is a set of coloops of $M|cl(A)$. In both cases, $|A_T \cap A| \geq 2$.

Let $a_T$ be the unique element of $T \cap cl(A)$. Let $T'$ be an element of $T - \{T\}$. By Lemma 2.9, $a_T \not\in T'$ and so $a_T \neq a_T'$.

We shall show next that

4.1.2. If $a_T \in T_i^* \cup T_j^*$, then $A_T \neq A_T'$, and $T_{ij}^* = T_{ij}^*$ for some $i$ and $j$ in $\{1, 2\}$.

Assume, without loss of generality, that $a_T \in T_2^*$. Then, by Lemma 2.9, $T_2^* = T_2^*$ for some $i$, say $i = 1$. Hence $T_1^* T_i^* T_j^* T_{ij}^* T$ is a chain. Moreover, either $T_1^* T_i^* T_j^* T_{ij}^* T, T_{ij}^* T$ is a chain, or $T_{ij}^* T = T_1^* T_2^* T_{ij}^* T_{ij}^* T$. In the first case, $A_T \neq A_T'$, as required. Now consider the second case, letting $X = T \cup T' \cup A_T'$. By (iii), $|E(M)| \geq 7$, so $|E(M) - X| \geq 1$. Thus, as $X - \{a_T, a_T'\}$ and $A_T$ span $X$ in $M$ and $M^*$, respectively, we have

$$1 \leq r(X) + r(X) - |X| \leq 4 + 3 - 6 = 1.$$  

Since $M$ is 3-connected, it follows that $|E(M)| = 7$ and $r(X) = 4$. But $r(T \cup T') = 3$ and $|E(M) - (T \cup T')| = 2$, so $M$ has a cocircuit of size at most two; a contradiction. We conclude that (4.1.2) holds.

It follows immediately from (4.1.2) that (ix)(a) holds. If $A_T \cap A_T' = \emptyset$, then

$$|(A_T \cup A_T') \cap A| = |A_T \cap A| + |A_T' \cap A| \geq 4.$$  

Hence (ix)(b) holds in this case, so we may suppose that $A_T \cap A_T' \neq \emptyset$. Let $S = A_T \cup A_T'$. Then, in $M|cl(A)$, the set $S$ is contained in a series class or is a set of coloops. Thus, by orthogonality and (a), $|A \cap S| \geq |S| - 1 \geq 3$. To complete the proof of (b), we now suppose that $|S \cap A| = 3$. Then $|S| = 4$, and $S$ must be contained in a series class of $M|cl(A)$ otherwise the unique element of $S - A$ is a coloop of $M|cl(A)$ contained in $cl(A) - A$; a contradiction. Thus $|A_T \cap A_T'| = 2$. But, by applying (4.1.2) to both $a_T$ and $a_T'$, we have that either $T_{ij}^* = T_{ij}^*$ for some $i$ and $j$, or $a_T \not\in A_T'$ and $a_T \not\in A_T$. In the latter case, $A_T - \{a_T\} = A_T - \{a_T\}$, that is, $A_T = T = A_T' = T'$ and we have established that (ix)(b) holds.

(x) Let $T'$ be the set of triangles $T$ of $M$ such that $T - cl(A) \neq \emptyset$. As $A$ does not span $M$, there is a cocircuit of $M$ that avoids $cl(A)$. By Theorem 2.8, for each such cocircuit $D$, there are distinct triangles $T_1'$ and $T_2'$ of $M$ such that $T'_i \cap D \neq \emptyset$. Hence $|T'| \geq 2$. 

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there is a cocircuit contained in $E(M) - \text{cl}(A)$ and this cocircuit has at least three elements, so $|E(M)| > 6$.

(iv) Since $T$ and $T^*$ meet but are distinct, orthogonality implies that $|T^* - T| = 1$. Let $a$ be the unique element of $T^* - T$. Suppose that $a \notin \text{cl}(A)$. Then $A$ is non-spanning and so, by (iii), $|E(M)| > 6$. Now $M \setminus a$ has $T \cap T^*$ as a 2-element cocircuit. Thus $\{T \cap T^*, E(M \setminus a) - (T \cap T^*)\}$ is a 2-separation of $M \setminus a$. But $T \cap T^*$ spans $T$ so $\{T, E(M \setminus a) - T\}$ is a non-minimal 2-separation of $M \setminus a$; a contradiction to (i). We conclude that $a \in \text{cl}(A)$.

(v) Assume that $M$ has a triad $T^*$ that avoids $A$. Then $|T^* \cap \text{cl}(A)| \leq 1$. Let $T^* = \{e_1, e_2, a_3\}$ and suppose that neither $e_1$ nor $e_2$ is in $\text{cl}(A)$. If $a_3 \notin \text{cl}(A)$, then $a_3 \in T^* \setminus \text{cl}(A) - A$. Hence, by Lemma 2.6, $T^* \cap A \neq \emptyset$; a contradiction. We may now assume that $a_3 \notin \text{cl}(A)$. Then $T^* \subseteq E(M) - \text{cl}(A)$. By the dual of Tutte's triangle lemma (2.4), there is a triangle $T$ of $M$ that meets the triad $T^*$ in exactly two elements. Thus, by (iv), the unique element of $T^* - T$ is in $\text{cl}(A)$. This is a contradiction since $T^* \cap \text{cl}(A) = \emptyset$.

(vi) Choose $N$ to be $M$ or $M^*$ so that both $T_1$ and $T_2$ are triangles of $N$. Assume that $|T_1 \cap T_2| = 2$. Then $N \setminus (T_1 \cup T_2)$ is isomorphic to $U_{2,1}$. Now take $e$ in $(T_1 \cup T_2) - \text{cl}(A)$. Then it is straightforward to check that $N \setminus e$ is 3-connected. Thus $M \setminus e$ or $M^* \setminus e$ is 3-connected, so either $M \setminus e$ or $M^* \setminus e$ is 3-connected; a contradiction to (ii).

(vii) Suppose that $T = \{f_1, f_2, b\}$ for some element $b$ and that, for some $i$, say $i = 1$, there is no triad meeting $\{f_1, f_2\}$ in $f_1$. By Tutte's triangle lemma (2.4), there is a triad $T^*$ that contains $f_1$ and exactly one of $b$ and $f_2$. Hence $b \notin T^*$ and $f_2 \in T^*$. By (vi), $T^*$ is the only triad of $M$ that contains $f_1$. By (i), the cosimplification $M'$ of $M \setminus f_1$ is 3-connected. As the only triad that contains $f_1$ is $T^*$, it follows that $M' = M \setminus f_1 / f_2$. But, as $|E(M)| \geq 5$, the matroid $M'$ is 3-connected having at least four elements and so is simple. Hence $M'$ equals the cosimplification $M''$ of $M / f_2$. Therefore $M''$ is 3-connected, which is contrary to (i).

(viii) Let $i$ and $j$ be distinct elements of $\{1,2,3\}$. Taking $f_1 = e_i$ and $f_2 = e_j$ in (vii), we deduce that $M$ has a triad $T_i^*$ such that $T_i^* \cap \{e_i, e_j\} = \emptyset$. Thus $T_i^* \cap T = T - e_i$. By (vi), $T_i^*$ is the unique such triad. Moreover, by (v), $T_i^*$ meets $A$ in a single element, $a_i$ say.

To finish the proof of (viii)(a), we need to check that $\{a_1, a_2, a_3\}$ is a triad of $M$. Observe that there is a cocircuit $D$ of $M$ such that $a_1 \in D \subseteq (T_1^* \cup T_2^*) - e_3 = \{a_1, a_2, e_1, e_2\}$. As $M$ is 3-connected, it follows that $|D| \geq 3$ and hence $e_i \in D$ for some $i$. By orthogonality, $e_1$ and $e_2$ must both belong to $D$, otherwise $D \cap T = \{e_i\}$. As $T_i^*$ is the unique triad such that $T_i^* \cap T = \{e_i, e_2\}$, it follows that $D = \{a_1, a_2, e_1, e_2\}$. Thus there is a cocircuit $D'$ of $M$ such that $a_1 \in D' \subseteq (D \cup T_3^*) - e_2 = \{a_1, a_2, a_3, e_1\}$. As $|D' \cap T| \neq 1$ and $|D'| \geq 3$, it follows that $e_1 \notin D'$ and that $D' = \{a_1, a_2, a_3\}$, that is, $\{a_1, a_2, a_3\}$ is a triad of $M$.

Part (b) is an immediate consequence of [11, Theorem 1.11]. Now we shall prove (c). Let $Z = \{a_1, a_2, a_3, e_1, e_2\}$. We start by proving the following.

4.1.1. Every circuit of $M(\{A \cup \{e_1, e_2\}\})$ that meets $Z$ must contain $Z$.

Let $C$ be such a circuit. Since each $a_i$ is a coloop of $M \setminus A$, the circuit $C$ must meet $\{e_1, e_2\}$. Moreover, as neither $e_1$ nor $e_2$ is spanned by $A$, it follows that $|C - A| \geq 2$ and hence $\{e_1, e_2\} \subseteq C$. As $|T_i^* \cap C| \neq 1$ for each $i$ in $\{1,2\}$, it follows that $a_i \in C$ for each such $i$. To prove that $Z \subseteq C$, it remains to show that $a_3 \in C$. Suppose that this is not the case. Then there is a circuit $C'$ of $M$ such
(iii) If \( A \) is not a spanning set of \( M \), then \(|E(M)| > 6 \) and \(|A| \geq 3\).
(iv) If \( T \) is a triangle and \( T^* \) is a triad of \( M \) such that \( T \neq T^* \) and \( T \cap T^* \neq \emptyset \), then \( T^* - T = \{a\} \) for some \( a \) in \( cl(A) \).
(v) Every triad of \( M \) meets \( A \).
(vi) Let \( T_1 \) and \( T_2 \) be distinct subsets of \( E(M) \) such that both are triangles or both are triads of \( M \). If \( (T_1 \cup T_2) - cl(A) \neq \emptyset \), then \( |T_1 \cap T_2| \leq 1 \).
(vii) If \( T \) is a triangle of \( M \) such that \( \{f_1, f_2\} \subseteq T - cl(A) \), then, for each \( i \) in \( \{1, 2\} \), there is a triad of \( M \) that meets \( \{f_1, f_2\} \) in \( \{f_i\} \).
(viii) If \( \{e_1, e_2, e_3\} \) is a triangle \( T \) of \( M \) and \( T \subseteq E(M) - cl(A) \), then, for each \( i \) in \( \{1, 2, 3\} \), there is a unique triad \( T_i^* \) of \( M \) such that \( T_i^* \cap T = T - e_i \). Moreover,
   (a) for each \( i \), the unique element \( a_i \) of \( T_i^* - T \) is in \( A \), and \( \{a_1, a_2, a_3\} \) is a triad of \( M \);
   (b) \( M^* = \Delta(M(K_4), M^* \setminus T) \) where \( \Delta = \{a_1, a_2, a_3\} \) and \( M(K_4) \) is labelled as in Figure 1, and \( M/T \) is 3-connected;
   (c) \( \{a_1, a_2, a_3\} \) is contained in a series class of \( (M/T)|_A \), and \( (M/T)|(A - a_i) = M'(A - a_i) \).
(ix) Let \( T \) be the set of triangles of \( M \) that meet both \( cl(A) \) and \( E(M) - cl(A) \). For every \( T \) in \( T \), there is a unique element \( a_T \) of \( T \cap cl(A) \) and there is a chain \( T_1^*, T_2^* \) whose rim \( A_T \) is contained in \( cl(A) \). Moreover, if \( T \) and \( T' \) are different elements of \( T \), then
   (a) \( A_T \neq A_{T'} \) and \( a_T \neq a_{T'} \);
   (b) \([A_T \cup A_{T'}] \cap A \geq 3 \) with equality only when (1) \( A_T \cup A_{T'} \) is contained in a series class of \( M|\cl(A) \); and (2) \( T_{ij}^* = T_{kl}^* \) for some \( i \) and \( j \) in \( \{1, 2\} \), or \( A_T - T = A_{T'} - T' \).
(x) If \( r(A) \leq 3 \) and \( A \) is not spanning, then \( M \) is isomorphic to the rank-four wheel having \( cl(A) \) as its rim.
(xi) If \( e \in E(M) - cl(A) \) such that \( M \) has no triangle that contains \( e \) and avoids \( cl(A) \), then \( e \) belongs to a triad that meets \( cl(A) \) in two elements.

Proof. (i) By Lemma 2.3, the required result follows if we can show that the simplification \( M'' \) of \( M/e \) is not 3-connected. Since \( e \) is a coloop of \( M|(A \cup e) \), we have \( (M/e)|A = M|A \). Therefore \( M'' \) can be labelled so that its ground set contains \( A \). Thus \( M'' \) cannot be 3-connected, otherwise \( M'' \) contradicts the minimality of \( (M, A) \).

(ii) If \( M \setminus e \) or \( M/e \) is 3-connected, then, since \( (M \setminus e)|A = M|A = (M/e)|A \), either \( (M \setminus e, A) \) or \( (M/e, A) \) contradicts the choice of \( (M, A) \). Thus neither \( M \setminus e \) nor \( M/e \) is 3-connected, so \( e \) is essential. Furthermore, by (i), the cosimplification of \( M \setminus e \) is 3-connected. Thus \( e \) must belong to a triad of \( M \).

(iii) When \(|A| = 1 \), we have that \( E(M) = A \); and when \(|A| = 2 \), we have that \( E(M) \) is a circuit of \( M \) having at most three elements. In both cases, \( A \) is spanning. Thus we may suppose that \(|A| \geq 3 \). If \(|A| = 3 \) and \( A \) is dependent, then \( A \) is either a circuit or a set of parallel elements, so \( A \) is spanning; a contradiction. Thus if \(|A| = 3 \), then \( A \) is independent. As \( A \) is non-spanning, it follows that \( r(M) \geq 4 \). Hence, since \( M \) is 3-connected, either \( r^*(M) \geq 3 \) and so \( |E(M)| > 6 \), or \( r^*(M) = 2 \). In the latter case, \( M \) is uniform and so, if \( e \in E(M) - cl(A) \), then \( M/e \) is also uniform of corank two, so \( M/e \) is 3-connected; a contradiction to (i). It remains to consider the case when \(|A| \geq 4 \). In that case, since \( A \) is non-spanning,
that $M_1 \setminus z_1, z_2$ is 3-connected and, repeating this argument, we eventually obtain that $M_1 \setminus z_1, z_2, \ldots, z_n$ is 3-connected. But the last matroid is $M^*$, hence $M$ is 3-connected. Let $M'$ be a minimal 3-connected minor of $M$ such that $M[A = M'|A]$. Now, for each element $e$ of $M^*$ that is not in $A$, there is another element $f$ of $E(M) - A$ such that $e$ and $f$ are both in the same attached $M(K_4)$ in $M_1$. Moreover, $e$ is in a triangle of $M^*$ with two elements of $A$ so $M^*/e$ has two elements of $A$ in parallel. Thus no 3-connected minor of $M^*/e$ contains all the elements of $A$. Since $M^*/e$ has a 2-cocircuit that contains $f$ and some element of $A$, a 3-connected minor of $M^*/e$ using all the elements of $A$ must be a 3-connected minor of $M^*\setminus e/f$. Since $f$ is also in a triangle of $M^*$ with two elements of $A$, we conclude no 3-connected minor of $M^*/e$ contains all the elements of $A$. Thus $(M, A)$ is indeed a minimal pair.

Now $|E(M)| = 3nk + 2n$ and $|A| = nk + 2n$. To find $M|A$, observe that this matroid is the dual of $M^*|A$. In the last matroid, there is a parallel class that contains $T_i - z_i$ together with all the elements of $A$ that are opposite $z_i$ in the copies of $M(K_4)$ which were attached across $T_i$. Moreover, by considering the $n$-raft $N$, it is not difficult to check that the simplification of $M^*|A$ is a circuit. Thus $M|A$ is the cycle matroid of the graph obtained by joining two vertices by $n$ internally disjoint paths each of length $k + 2$. Therefore $\lambda_1 (A, M) = 1$, $\lambda_2 (A, M) = (k + 1)n$, $\lambda_3 (A, M) = (k - 1)n$, and $\alpha (A, M) = 1$. It follows easily that equality holds in Theorem 1.5. One can easily check that the bound in Theorem 1.6 exceeds $|E(M)|$ by exactly one in this case. Moreover, $|E(M)|$ exceeds the bound in Theorem 1.4 by $(k - 1)n - 3$, which can be arbitrarily large.

To close this section, we observe that $U_{2,6}$ is a 2-raft. If we perform the above construction beginning with this matroid, we obtain a matroid $M$ for which $M|A$ is independent and equality is attained in Theorem 1.3.

4. Minimal pairs

The purpose of this section is to prove numerous properties of minimal pairs that will be used in the proofs of the main results. The latter will be given in Section 5. We begin with a lemma that gathers together eleven such properties.

4.1. Lemma. A minimal pair $(M, A)$ has the following properties:

(i) Suppose $e \in E(M) - cl(A)$. Then $M\setminus e$ has no non-minimal 2-separations and the cosimplification of this matroid is 3-connected. Moreover, the simplification of $M/e$ is not 3-connected.

(ii) If $e \in E(M) - cl(A)$, then $e$ is essential and belongs to a triad of $M$.
The last lemma was used in the proof of the following theorem, the main result of [5]

2.8. Theorem. Let $C^*$ be a cocircuit of a 3-connected matroid $M$ and suppose that, for every element $e$ of $C^*$, the contraction $M/e$ is not 3-connected. Then $M$ has triangles $T_1$ and $T_2$ such that $T_1 \cap C^*$ and $T_2 \cap C^*$ are distinct and non-empty.

Next we note a useful property of chains of triangles and triads [11, Lemma 3.4].

2.9. Lemma. Let $e_1, e_2, e_3, e_4, e_5$ be distinct elements of a 3-connected matroid $M$ that is not isomorphic to $M(K_4)$. Suppose that $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triangles and $\{e_2, e_3, e_4\}$ is a triad of $M$. Then these two triangles and this one triad are the only triangles and triads of $M$ containing $e_3$.

3. Extremal examples

It was noted in the introduction that all the bounds in the theorems there are sharp. In this section, we describe infinite families of examples that attain equality in Theorems 1.5 and 1.3.

The examples given here will use the operation of generalized parallel connection [2]. Suppose that the intersection of the ground sets of the matroids $M$ and $M(K_4)$ is $\Delta$ and that $\Delta$ is a triangle in both matroids. The generalized parallel connection of $M(K_4)$ and $M$ across $\Delta$ is the matroid $P_\Delta(M(K_4), M)$ whose ground set is the union of the ground set of the two matroids and whose flats are the subsets $X$ of the ground set so that $X \cap E(M(K_4))$ is a flat of $M(K_4)$ and $X \cap E(M)$ is a flat of $M$. If the elements of $\Delta$ are deleted from $P_\Delta(M(K_4), M)$, we obtain the same matroid that we would get by performing a $\Delta - Y$-exchange on $M$ across $\Delta$.

Let $n$ be an integer exceeding one. An $n$-raft [4] is a matroid of rank $2n - 2$ whose ground set is the union of $n$ disjoint triangles such that, for all $m < n$, the union of every set of $m$ of these triangles has rank $2m$. One example of an $n$-raft is the matroid $M^*(K_3, n)$. Another is the matroid that is obtained from the direct sum of two $n$-element circuits $\{x_1, x_2, \ldots, x_n\}$ and $\{z_1, z_2, \ldots, z_n\}$ by, for each $i$ in $\{1, 2, \ldots, n\}$, freely adding a new element $y_i$ on the line joining $x_i$ and $z_i$.

We construct our family of examples by beginning with an $n$-raft $N$ for some $n \geq 3$. Let the distinguished triangles of the raft be $T_1, T_2, \ldots, T_n$ where $T_i = \{x_i, y_i, z_i\}$ for all $i$, and assume that the raft has the additional property that $\{z_1, z_2, \ldots, z_n\}$ is a circuit. Let $k$ be a positive integer. By repeated generalized parallel connections, attach exactly $k$ distinct copies of $M(K_4)$ across each $T_i$ in $N$. Let the resulting matroid be $M_1$. Now, for each $i$, delete $z_i$ from $M_1$. Take $M$ to be the dual of the resulting matroid. In each copy of $M(K_4)$ that was attached across some $T_i$, pick the opposite element to $z_i$, that is, the element of the $M(K_4)$ that is not in a triangle with $z_i$. Do this for all $i$ and let $A$ consist of these $nk$ elements together with the $2n$ elements of $N$ that remain in $M$.

We assert that $(M, A)$ is a minimal pair. First note that $N$ is certainly connected. Moreover, if $\{X, Y\}$ is a 2-separation of $N$, then neither $X$ nor $Y$ spans $N$. It is easy to see that each of $X$ and $Y$ may be assumed to be a union of distinguished triangles and from this we obtain a contradiction to the fact that $\{X, Y\}$ is a 2-separation. Thus $N$ is 3-connected. As $M(K_4)$ is also 3-connected, so too is $M_1$ [10]. Moreover, it is clear that $M_1 / z_1$ does not simplify to a 3-connected matroid. Thus the cosimplification of $M_1 / z_1$ is 3-connected. This cosimplification is equal to $M_1 \setminus z_1$ since $z_1$ is in no triads of $M_1$. By a similar argument, we obtain
$T_1, T_2, \ldots, T_k$ is maximal, then the elements can be labelled so that neither $e_1$ nor $e_k+2$ is essential.

Let $\{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}, \ldots, \{e_k, e_{k+1}, e_{k+2}\}$ be a chain for some $k \geq 3$. Partition $\{e_1, e_2, \ldots, e_{k+2}\}$ into $X_e$, those $e_i$ for which $i$ is odd, and $X_o$, those $e_i$ for which $i$ is odd. If $\{e_1, e_2, e_3\}$ is a triad, then $X_o$ is the rim of the chain and $X_e$ is its set of spokes. If $\{e_1, e_2, e_3\}$ is a triangle, then $X_e$ is its rim and $X_o$ is its set of spokes.

In a 3-connected matroid other than a wheel or whirl, a maximal chain is called a fan. A fan has type-1, type-2, or type-3 if its first and last links consist of, respectively, two triangles, two triads, or one triangle and one triad. Oxley and Wu [11] proved the following result, which will be very useful here.

2.2. Theorem. Let $M$ be a 3-connected matroid that is neither a wheel nor a whirl. Suppose that $e$ is an essential element of $M$. Then $e$ is in a fan. Moreover, this fan is unique unless

(a) every fan containing $e$ consists of a single triangle and any two such triangles meet in $\{e\}$;
(b) every fan containing $e$ consists of a single triad and any two such triads meet in $\{e\}$; or
(c) $e$ is in exactly three fans; these three fans are of the same type, each has five elements, together they contain a total of six elements, and, depending on whether these fans are of type-1 or type-2, the restriction or contraction, respectively, of $M$ to this set of six elements is isomorphic to $M(K_4)$.

The following two basic results will be used repeatedly throughout the paper. The first is due to Bixby [1]. The second is Tutte’s triangle lemma [13] (see also [9, Lemma 8.4.9]).

2.3. Lemma. Let $e$ be an element of a 3-connected matroid $M$. Then either $M\setminus e$ or $M/e$ has no non-minimal 2-separations. Moreover, in the first case, the cosimplification of $M\setminus e$ is 3-connected, while in the second case, the simplification of $M/e$ is 3-connected.

2.4. Lemma. Let $M$ be a 3-connected matroid having at least four elements and suppose that $\{e, f, g\}$ is a triangle of $M$ such that neither $M\setminus e$ nor $M\setminus f$ is 3-connected. Then $M$ has a triad that contains $e$ and exactly one of $f$ and $g$.

The elementary proof of the next lemma is omitted.

2.5. Lemma. If $\{Z, W\}$ is a $k$-separation of a matroid $M$ and $r(Z) = |Z|$, then every $k$-element subset of $Z$ contains a cocircuit of $M$.

The next lemma is a straightforward consequence of orthogonality, the property of a matroid that a circuit and a cocircuit cannot have exactly one common element.

2.6. Lemma. If $D$ is a cocircuit of a matroid $M$ such that $D \cap (\text{cl}(A) - A) \neq \emptyset$, then $D \cap A \neq \emptyset$. In particular, $|D \cap \text{cl}(A)| \geq 2$.

The following lemma is proved in [5, (2.12)].

2.7. Lemma. Let $e$ and $f$ be distinct elements of a 3-connected matroid $M$ such that $M\setminus e$ is not 3-connected. If $M$ has triangles $T$ and $T'$ containing $e$ and $f$, respectively, such that $|T \cap T'| = 1$ and $T \cup f$ is a cocircuit of $M$, then $e$ is in a triad of $M$. 
3, we shall present a family of extremal examples for Theorem 1.5 in each member of which, $M[A]$ can be chosen to have many series classes of large rank. Prior to that, Section 2 introduces notation, terminology, and some important known results that will be needed. Section 4 develops the properties of minimal pairs that will be used in the proofs of the main results, and these proofs will be given in Section 5.

The essential difference between the bounds in Theorems 1.2 and 1.5 is the presence of $\lambda_2(A, M)$ in the latter. But the former assumes that $A$ spans $M$. Indeed, if $A$ is non-spanning and $(M, A)$ is a minimal pair, then, for a basis $X$ of $M/A$, clearly $(M, A \cup X)$ is a minimal pair in which $A \cup X$ is spanning. Hence Theorem 1.2 can be applied to this minimal pair. This crude technique yields the bound

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - 2 + 3(r(M) - r(A)).$$

In the next theorem, our second main result, we shall show that this bound can be sharpened.

**1.6. Theorem.** If $(M, A)$ is a minimal pair, then

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + r(M) - r(A) - \alpha(A, M),$$

where

$$\alpha(A, M) = \begin{cases} 0, & \text{when } A \text{ is a circuit of } M \\ 1, & \text{when } A \text{ is not a circuit} \end{cases}$$

If $M$ is an $n$-spoked wheel with $n \geq 4$ and $A$ is its rim, then $(M, A)$ is a minimal pair for which equality is attained in the last theorem. However, for this example, the difference between the bound in Theorem 1.5 and $|E(M)|$ is $n - 4$.

## 2. Preliminaries

In this section, we note a number of results that will be used in the proofs of the main theorems. We shall follow Oxley [9] for basic notation and terminology. We shall require some additional terminology and results relating to 3-connected matroids. Tutte [13] calls an element $e$ of a 3-connected matroid $M$ **essential** if neither the deletion $M\setminus e$ nor the contraction $M/e$ remains 3-connected. Tutte showed that every essential element in a 3-connected matroid is in a triangle or a triad. Indeed, triangles and triads appear constantly in the study of 3-connected matroids, and a very useful concept in this study is that of a chain of triangles and triads. Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid $M$ such that, for all $i$ in \{1, 2, \ldots, $k - 1$\},

(i) $\{T_i, T_{i+1}\}$ contains exactly one triangle and exactly one triad;
(ii) $|T_i \cap T_{i+1}| = 2$; and
(iii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \ldots \cup T_i)$ is empty.

Then we call $T_1, T_2, \ldots, T_k$ a **chain of $M$ of length $k$ with links $T_1, T_2, \ldots, T_k$**. By extending the proof of Tutte’s Wheels and Whirls Theorem [13], Oxley and Wu [11] showed that if such a chain is maximal, then the elements at both ends are non-essential. In particular, they proved the following.

**2.1. Lemma.** Let $T_1, T_2, \ldots, T_k$ be a chain in a 3-connected matroid $M$. Then $M$ has $k + 2$ distinct elements $e_1, e_2, \ldots, e_{k+2}$ such that $T_i = \{e_i, e_{i+1}, e_{i+2}\}$ for all $i$. Suppose that $|E(M)| \geq 4$ and $M$ is not a wheel or a whirl. If the chain
unless \( A \) is a circuit of \( M \) of size at least four, in which case,
\[
|E(M)| \leq 2|A| - 2.
\]

It is also shown in [6] that the bounds in Theorem 1.2 are sharp. Indeed, examples are given that attain the bounds for all \( A \) such that \( M|A \) is simple but not free.

In this paper, we shall prove several results. We solve Problem 1.1 when \( N \) is a free matroid by proving the following result.

1.3. \textbf{Theorem.} Let \( M \) be a 3-connected matroid and let \( A \) be an independent set of \( M \). If \( M \) has no proper 3-connected minor \( M' \) in which \( A \) is independent, then \( |E(M)| \leq 3|A| - 1 \).

Note that the bound in this theorem is sharp. To see this, suppose that \( n \geq 3 \) and let \( K_{3,n}'' \) be the graph obtained from \( K_{3,n} \) by joining a degree-\( n \) vertex \( v \) of the latter to the two vertices to which it is non-adjacent. Then it is not difficult to check that equality is attained in the theorem if we take \( M \) to be \( M^*(K_{3,n}'') \) and \( A \) to be any set consisting of all but one of the edges meeting \( v \). When \( A \) is both independent and spanning, it is shown in [7] that the bound in Theorem 1.3 can be improved by 3.

If \( A \) is a subset of the ground set of a 3-connected matroid \( M \), then \( (M, A) \) is a \textit{minimal pair} if \( M \) has no proper 3-connected minor \( M' \) such that \( M'|A = M|A \). Thus, in both Theorems 1.2 and 1.3, \( (M, A) \) is a minimal pair. Our second theorem shows that the bound in Theorem 1.2 also holds when we omit the hypothesis that \( A \) is spanning, provided \( A \) has no small cocircuits. More specifically:

1.4. \textbf{Theorem.} Let \( (M, A) \) be a minimal pair such that \( M|A \) has no coloops, every series class of \( M|A \) has rank at most two, and \( M \not\cong U_{1,3} \). Then
\[
|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - 2.
\]

The bound in the last result is sharp since, as noted above, the bounds in Theorem 1.2 are sharp. We shall show in Section 3 that if there no restriction on the rank of the series classes of \( M|A \), then \( |E(M)| \) can exceed the bound in the last result by an arbitrarily large number. These examples prompt the introduction of another function, \( \lambda_3(A, M) \), which is defined by
\[
\lambda_3(A, M) = \sum_S (|S| - 3),
\]
where the sum is taken over all series classes \( S \) of \( M|A \) with at least four elements. The following theorem is one of the two main results of the paper.

1.5. \textbf{Theorem.} If \( (M, A) \) is a minimal pair, then
\[
|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) + \lambda_3(A, M) - \alpha(A, M),
\]
where
\[
\alpha(A, M) = \begin{cases} 
0, & \text{when } A \text{ is a circuit of } M \\
1, & \text{when } A \text{ is not a circuit}
\end{cases}
\]

Note that Theorem 1.3 will follow immediately from the last theorem for, when \( A \) is independent, \( \lambda_1(A, M) = \lambda_2(A, M) = |A|, \lambda_3(A, M) = 0, \) and \( \alpha(A, M) = 1 \). Although the function \( \lambda_3(A, M) \) may look strange, it is indispensable, even for cographic matroids. For example, for \( n \geq 3 \), let \( M \) be \( M^*(K_{3,n}''') \) and let \( A \) be the set of edges meeting the degree-(\( n + 2 \)) vertex of \( K_{3,n}''' \). Then \( M|A \) is a circuit and it is straightforward to check that equality is attained in Theorem 1.5. In Section
ON THE 3-CONNECTED MATROIDS THAT ARE MINIMAL HAVING A FIXED RESTRICTION

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Abstract. Let $N$ be a restriction of a 3-connected matroid $M$ and let $M'$ be a 3-connected minor of $M$ that is minimal having $N$ as a restriction. This paper gives a best-possible upper bound on $|E(M') - E(N)|$.

1. Introduction

If $X$ is a subset of the ground set of a 3-connected matroid $M$, what can be said about the size of a minimal 3-connected minor $M'$ of $M$ that not only includes $X$ in its ground set but also maintains the matroid structure on $X$? For instance, if $X$ is a circuit, a basis, or an independent set of $M$, then $X$ is a circuit, a basis, or an independent set, respectively, of $M'$. The purpose of this paper is to answer this question. More specifically, we solve the following:

1.1. Problem. Let $N$ be a restriction of a 3-connected matroid $M$ and let $M'$ be a 3-connected minor of $M$ that is minimal having $N$ as a restriction. Give a sharp upper bound on $|E(M') - E(N)|$.

The natural modification of this problem in which “restriction” is replaced by “minor” seems much more difficult, although we do hope to return to it in future work. For this modified problem, Truemper [12] proved, under the additional constraints that $N$ is 3-connected but different from $M'$, that $|E(M') - E(N)| \leq 3$. In certain natural cases, including those raised above, when $N$ is a circuit, a basis, or an independent set of $M$, the modified and original problems coincide.

Let $M$ be a matroid and $A$ be a subset of $E(M)$. We define $\lambda_1(A, M)$ to be the number of connected components of $M|A$. Now $M|A$ can be constructed from a collection $\lambda_2(A, M)$ of 3-connected matroids by using the operations of direct sum and 2-sum. It follows from results of Cunningham and Edmonds (see Cunningham [3]) that $\lambda_2(A, M)$ is unique up to isomorphism. We denote by $\lambda_2(A, M)$ the number of matroids in $\lambda_2(A, M)$ that are not isomorphic to $U_{1,3}$, the three-element cocircuit.

The next theorem, the main result of [6], solves Problem 1.1 in the case when $N$ spans $M$.

1.2. Theorem. Let $M$ be a 3-connected matroid other than $U_{1,3}$ and let $A$ be a non-empty spanning subset of $E(M)$. If $M$ has no proper 3-connected minor $M'$ such that $M'|A = M|A$, then

$$|E(M)| \leq |A| + \lambda_1(A, M) + \lambda_2(A, M) - 2,$$

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