Packing Cycles in Graphs

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Abstract

A graph $G$ is called cycle Mengerian (CM) if for any nonnegative integral function $w$ defined on $V(G)$, the maximum number of cycles in $G$ such that each vertex $v$ is used at most $w(v)$ times is equal to the minimum of $\sum\{w(v) : v \in S\}$, where the minimum is taken over all $S \subseteq V(G)$ such that deleting $S$ from $G$ results a forest. The purpose of this paper is to characterize all CM graphs in terms of forbidden structures.
1 Introduction

Graphs considered in this paper are finite, simple, and undirected. Let $G = (V,E)$ be a graph with nonnegative integral weight $w(v)$ on each $v \in V$. A collection $\mathcal{C}$ of cycles of $G$ is called a cycle packing if each vertex $v$ of $G$ is used at most $w(v)$ times by members of $\mathcal{C}$; a set $X$ of vertices in $G$ is called a feedback set if deleting $X$ from $G$ results a forest. The cycle packing problem is to find a cycle packing with maximum size, and the feedback set problem is to find a feedback set with minimum total weight; both problems are NP-hard [4] and the latter arises in a variety of applications. It is clear that

$$\max\{|\mathcal{C}|: \mathcal{C} \text{ is a cycle packing}\} \leq \min\{\sum_{v \in X} w(v) : X \text{ is a feedback set}\}.$$ 

However, the minimax equality need not hold in general; in fact, the ratio of the two sides can be arbitrarily large even when $w(v) = 1$ for all vertices $v \in V$, as shown by Erdős and Pósa [3]. We shall call $G$ cycle Mengerian (CM) if the above inequality holds with equality for all $w$. The purpose of this paper is to characterize all CM graphs in terms of forbidden structures.

Let us define some graphs before presenting our theorem. A $\Theta$-graph is a subdivision of $K_{2,3}$. A wheel is obtained from a cycle by adding a new vertex and making it adjacent to all vertices of the cycle. A $W$-graph is a subdivision of a wheel. An odd ring is a graph obtained from an odd cycle by replacing each edge $e = xy$ with either a cycle containing $e$ or two triangles $xab$, $yed$ together with two vertex-disjoint paths between $\{a,b\}$ and $\{c,d\}$.

![Figure 1. An odd ring, where the thin edges could be subdivided.](attachment:image.png)

For convenience, we shall simply say that a graph $G$ has a graph $H$ if $H$ is isomorphic to an induced subgraph of $G$.

**Theorem 1.1.** A graph is CM if and only if it has no $\Theta$-graphs, nor $W$-graphs, nor odd rings.
Notice that some $W$-graphs may have $\Theta$-graphs or other $W$-graphs. We present Theorem 1.1 in the current form so as to make the statement cleaner. It is not difficult to verify that the minimal none-CM $W$-graphs are those obtained from a wheel by subdividing the rim edges, and those obtained from $K_4$ by subdividing each of the three edges in a star at least once.

To prove the theorem, we introduce a partition property in section 2, which is sufficient for a graph to be CM. We prove that, when piecing together graphs with this partition property, the resulting graph also has the property. In section 3, we derive a structural theorem, which asserts that if a graph has no forbidden structures, then it can be expressed as “sums” of some prime graphs. In section 4, we show that every prime graph enjoys the partition property, which, together with the results established in section 2, yields our main theorem.

2 Sums of hypergraphs

As outlined in the last section, the basic idea underlying our proof is to express CM graphs as sums of some prime graphs. The purpose of this section is to derive some results concerning summing operations. We shall state these results in terms of hypergraphs, since the more general form may have potential applications elsewhere and since the proofs are easier to describe in this way.

A hypergraph is simply a collection $\Gamma$ of subsets of a finite set $V$. Members of $V$ and $\Gamma$ are called vertices and hyperedges, respectively. For a nonnegative integral function $w$ on $V$, a $w$-matching of $\Gamma$ is a collection $M$ of hyperedges (repetition is allowed) such that each vertex $x$ in $V$ is used at most $w(x)$ times by members of $M$. The maximum of $|M|$, over all $w$-matchings of $\Gamma$, is denoted by $\nu_w(\Gamma)$. A transversal of $\Gamma$ is a minimal (under inclusion) set $T$ of vertices such that $T \cap A \neq \emptyset$ for all members $A$ of $\Gamma$. We denote by $\tau_w(\Gamma)$ the minimum of $\sum\{w(x) : x \in T\}$, where the minimum is taken over all transversals $T$ of $\Gamma$. Clearly,

$$\nu_w(\Gamma) \leq \tau_w(\Gamma)$$

for all $\Gamma$ and $w$. If $\Gamma$ is a hypergraph such that the above inequality holds with equality for all $w$, then $\Gamma$ is called Mengerian [8]. With this terminology, we can see that a graph $G$ is CM if and only if $\Gamma_G$ is Mengerian, where $\Gamma_G$ is the cycle hypergraph of $G$ which consists of the vertex-sets of all cycles of $G$.

People familiar with integer programming may like the following equivalent definition of Mengerian hypergraphs. Let $M$ be the hyperedge-vertex incidence matrix of $\Gamma$. Then $\Gamma$ is Mengerian if and only if the linear system \{$Mx \geq e, x \geq 0$\} is TDI [2, 7], where $e$ is the all-one vector. This, by the Edmonds-Giles theorem [2, 7], amounts to that both of the following two
problems

\[
\begin{align*}
\max & \quad y^T e \\
\text{s.t.} & \quad y^T M \leq w^T \\
\min & \quad w^T x \\
\text{s.t.} & \quad M x \geq e \\
y & \geq 0 \\
x & \geq 0
\end{align*}
\]

have integral optimal solutions, for all nonnegative integral vectors \( w \).

Usually it is very difficult to recognize Mengerian hypergraphs by using the above definitions. In the following, we introduce a property, which is sufficient for a hypergraph to be Mengerian and is much easier to work with. Let \( \Gamma \) be a hypergraph with vertex set \( V \). For any collection \( \Lambda \) of members of \( \Gamma \), we shall let \( d_\Lambda(x) \) denote the number of hyperedges in \( \Lambda \) that contain \( x \). For any \textbf{subset} \( \Lambda \) of \( \Gamma \), a \textbf{subpartition} of \( \Lambda \) consists of two collections \( \Lambda_1 \) and \( \Lambda_2 \) of members of \( \Gamma \) (which are not necessarily in \( \Lambda \)) such that

(i) \( |\Lambda_1| + |\Lambda_2| = |\Lambda| \),
(ii) \( d_{\Lambda_1}(x) + d_{\Lambda_2}(x) \leq d_\Lambda(x) \) for all \( x \) in \( V \), and
(iii) Each member of \( \Lambda \) with size 3 is contained in \( \Lambda_1 \cup \Lambda_2 \).

We remark here that repetition is allowed in both \( \Lambda_1 \) and \( \Lambda_2 \), but not allowed in \( \Lambda \). The subpartition \( \Lambda_1, \Lambda_2 \) is called \textbf{equitable} if \( \max\{d_{\Lambda_1}(x), d_{\Lambda_2}(x)\} \leq \lfloor d_\Lambda(x)/2 \rfloor \) for all \( x \) in \( V \). The hypergraph \( \Gamma \) is called \textbf{equitably subpartitionable} (ESP) if every subset \( \Lambda \) of \( \Gamma \) admits an equitable subpartition.

**Theorem 2.1.** Every ESP hypergraph is Mengerian.

To prove this theorem, we need the following result of Lovász [5, 6].

**Lovász’ Theorem.** A hypergraph \( \Gamma \) is Mengerian if and only if \( \nu_{2w}(\Gamma) \leq 2\nu_w(\Gamma) \) for all nonnegative integral functions \( w \).

**Proof of Theorem 2.1.** Let \( \Gamma \) be an ESP hypergraph on \( V \). To show that \( \Gamma \) is Mengerian, by Lovász’ theorem, we may turn to verify that \( \nu_{2w}(\Gamma) \leq 2\nu_w(\Gamma) \) for any nonnegative integral function \( w \) defined on \( V \). We prove this by finding a \( w \)-matching of size at least \( \nu_{2w}(\Gamma)/2 \).

Let \( M \) be a \( 2w \)-matching of \( \Gamma \) of size \( \nu_{2w}(\Gamma) \) and, for each hyperedge \( A \) of \( \Gamma \), let \( M(A) \) be the number of times that \( A \) appears in \( M \). Let \( \Lambda \) be the set of hyperedges \( A \) with \( M(A) \) odd. Since \( \Gamma \) is ESP, \( \Lambda \) admits an equitable subpartition \( (\Lambda_1, \Lambda_2) \). Let \( M_0 \) be a collection of hyperedges such that each \( A \) appears \( \lfloor M(A)/2 \rfloor \) times. Set \( M_i = M_0 \cup \Lambda_i \) \((i = 1, 2)\). Clearly \( |M_1| + |M_2| = |M| = \nu_{2w}(\Gamma) \), and both \( M_1 \) and \( M_2 \) are \( w \)-matchings. It follows that at least one of these \( w \)-matchings has size at least \( \nu_{2w}(\Gamma)/2 \). \( \square \)

A graph \( G \) is called \textbf{ESP} if its cycle hypergraph \( \Gamma_G \) is ESP. The next corollary follows obviously from Theorem 2.1. We point out that all corollaries in this section can be deduced easily from the corresponding theorems by considering the cycle hypergraph.
Corollary 2.1. Every ESP graph is CM.

It was proved by Seymour [8] that all “minors” of a Mengerian hypergraph are Mengerian. We do not need this fact in our paper, but we do need a weaker version of it. Let \( \Gamma \) be a hypergraph on \( V \) and let \( U \subseteq V \). Then \( \Gamma \setminus U \) is the hypergraph on \( V \setminus U \) with hyperedges all \( A \in \Gamma \) for which \( A \cap U = \emptyset \). If \( U \) consists of a single vertex \( u \), we shall write \( \Gamma \setminus u \) instead of \( \Gamma \setminus \{u\} \). The next proposition says that being Mengerian is preserved under deleting vertices.

Theorem 2.2. If \( \Gamma \) is a Mengerian hypergraph on \( V \) and \( U \subseteq V \), then \( \Gamma \setminus U \) is also Mengerian.

Proof. For any nonnegative integral function \( w \) on \( V \setminus U \), let us define \( w^+ \) on \( V \) with \( w^+(x) = 0 \) for all \( x \in U \) and \( w^+(x) = w(x) \) for all \( x \in V \setminus U \). Then it is straightforward to verify that
\[
\nu_w(\Gamma \setminus U) = \nu_{w^+}(\Gamma) = \tau_{w^+}(\Gamma) = \tau_w(\Gamma \setminus U).
\]

Throughout this paper, for any vertex-set or edge-set \( Z \) of a graph \( G \), we denote by \( G \setminus Z \) the graph obtained from \( G \) by deleting \( Z \); when \( Z \) is a singleton \( \{z\} \), we write \( G \setminus z \) for short.

Corollary 2.2. If \( G \) is CM and \( U \subseteq V(G) \), then \( G \setminus U \) is also CM.

Theorem 2.3. Suppose \( \Gamma \) is obtained by identifying \( k \) vertices of \( \Gamma_1 \) with \( k \) vertices of \( \Gamma_2 \) \((k = 0, 1)\). If both \( \Gamma_1 \) and \( \Gamma_2 \) are ESP, then so is \( \Gamma \).

Proof. Let \( \Lambda \) be a subset of \( \Gamma \), let \( \Lambda_1 = \Lambda \cap \Gamma_1 \), and let \( \Lambda_2 = \Lambda - \Lambda_1 \). Since each \( \Gamma_i \) is ESP, \( \Lambda_i \) has an equitable subpartition \((\Lambda_1^1, \Lambda_1^2)\). Clearly, \( S = (\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1) \) is a subpartition of \( \Lambda \). Moreover, when \( k = 0 \), it is also clear that \( S \) is equitable. When \( k = 1 \), let \( x \) be the common vertex of \( \Gamma_1 \) and \( \Gamma_2 \). Without loss of generality, let \( d_{\Lambda_1^1}(x) \leq d_{\Lambda_1^2}(x) \) for \( i = 1, 2 \). Then it is easy to see that \( S \) is equitable.

Let \( G_1 \) and \( G_2 \) be two graphs. The \( 0 \)-sum of \( G_1 \) and \( G_2 \) is obtained by taking the disjoint union of \( G_1 \) and \( G_2 \); the \( 1 \)-sum is obtained by identifying a vertex of \( G_1 \) with a vertex of \( G_2 \). The following corollary follows instantly from Theorem 2.3.

Corollary 2.3. Suppose \( G \) is the \( 0 \)- or \( 1 \)-sum of \( G_1 \) and \( G_2 \). If both \( G_1 \) and \( G_2 \) are ESP, then so is \( G \).

Theorem 2.4. Let \( \Gamma \) be obtained by identifying vertices \( x_1, y_1 \) of \( \Gamma_1 \) with vertices \( x_2, y_2 \) of \( \Gamma_2 \). For \( i = 1, 2 \), let \( \Gamma'_i \) be obtained from \( \Gamma_i \) by adding a new vertex \( z_i \) and a new edge \( \{x_i, y_i, z_i\} \). If both \( \Gamma'_1 \) and \( \Gamma'_2 \) are ESP, then so is \( \Gamma \).

Proof. Let \( \Lambda \) be a set of hyperedges of \( \Gamma \). We need to find an equitable subpartition of \( \Lambda \). Set \( \Lambda_1 = \Lambda \cap \Gamma_1 \) and \( \Lambda_2 = \Lambda - \Lambda_1 \). We consider the following two cases.

Case 1. At least one of \( d_{\Lambda_1}(v_i) \), for \( v_i \in \{x_i, y_i\} \) and \( i \in \{1, 2\} \), is even. Let us assume that
$d_{\lambda_i}(x_1)$ is even. Since each $\Gamma'_i$ is ESP, $\Lambda_i$ has an equitable subpartition ($\Lambda_1^1, \Lambda_2^2$). Without loss of generality, for $i = 1, 2$, let us assume that $d_{\lambda_i'}(y_i) \leq d_{\lambda_2}(y_i)$. Then it is straightforward to verify that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^2)$ is an equitable subpartition of $\Lambda$.

Case 2. $d_{\lambda_i}(v_i)$ is odd, for all $v_i \in \{x_i, y_i\}$ and $i \in \{1, 2\}$. Let $A_i = \{x_i, y_i, z_i\}$. Since each $\Gamma'_i$ is ESP, $A_i \cup \{A_i\}$ admits an equitable subpartition. As $|A_i| = 3$ and $d_{\lambda_i(\{A_i\})}(z_i) = 1$, from definition we conclude that $A_i$ appears precisely once in this subpartition. Let $(\Lambda_1^1 \cup \{A_i\}, \Lambda_2^2)$ denote this subpartition. Since $A_i$ is the only hyperedge containing $z_i$ in $\Gamma'_i$, each member of $\Lambda_1^1 \cup \Lambda_2^2$ is in $\Gamma_i$. Now it is straightforward to verify that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^2)$ is an equitable subpartition of $\Lambda$.

The 2-sum of two graphs $G_1$ and $G_2$ is obtained by first choosing a triangle $x_i,y_i,z_i$ from $G_i$ ($i = 1, 2$) such that $z_i$ has degree two in $G_i$, then deleting $z_i$ from $G_i$ ($i = 1, 2$), and finally, identifying $x_1y_1$ with $x_2y_2$. As a corollary of Theorem 2.4, we have the following statement on 2-sum operation.

Corollary 2.4. Suppose $G$ is a 2-sum of $G_1$ and $G_2$. If both $G_1$ and $G_2$ are ESP, then so is $G$.

Theorem 2.5. Let $B_i = \{x_{i1}, x_{i2}, x_{i3}\}$ ($i = 1, 2$) be an edge of $\Gamma_i$ and let $\Gamma$ be obtained by identifying $x_{ij}$ with $x_{2j}$ ($j = 1, 2, 3$). For $i = 1, 2$ and $1 \leq j < k \leq 3$, let $\Gamma_{ijk}$ be obtained from $\Gamma_i$ by adding a new vertex $x_{ijk}$ and a new edge $A_{ijk} = \{x_{ijk}, x_{ij}, x_{ik}\}$. If all $\Gamma_{ijk}$ are ESP, then so is $\Gamma$.

Proof. Let $\Lambda$ be a set of hyperedges of $\Gamma$. We need to find an equitable subpartition of $\Lambda$. Set $\Lambda_1 = \Lambda \cap \Gamma_1$ and $\Lambda_2 = \Lambda - \Lambda_1$. We consider the following three cases.

Case 1. At least two of the three sets $\{d_{\lambda_1}(x_{1j}), d_{\lambda_2}(x_{2j})\}$, for $j = 1, 2, 3$, contain even numbers. Let us assume that $\{d_{\lambda_1}(x_{11}), d_{\lambda_2}(x_{21})\}$ and $\{d_{\lambda_1}(x_{12}), d_{\lambda_2}(x_{22})\}$ contain even numbers. Since each $\Gamma_{i12}$ is ESP, $\Lambda_i$ has an equitable subpartition ($\Lambda_1^1, \Lambda_2^2$). Without loss of generality, for $i = 1, 2$, let us assume that $d_{\lambda_i^1}(x_{i3}) \leq d_{\lambda_i^2}(x_{i3})$. Then it is straightforward to verify that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^2)$ is an equitable subpartition of $\Lambda$.

Case 2. Exactly one of the three sets $\{d_{\lambda_1}(x_{1j}), d_{\lambda_2}(x_{2j})\}$ ($j = 1, 2, 3$), say $j = 3$, contains even numbers. Since each $\Gamma_{i12}$ is ESP, like in Case 2 of the proof of Theorem 2.4, $\Lambda_i \cup \{A_{i12}\}$ has an equitable subpartition ($\Lambda_1^1 \cup \{A_{i12}\}, \Lambda_2^2$). Then it is straightforward to verify that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^2)$ is an equitable subpartition of $\Lambda$.

Case 3. $d_{\lambda_i}(x_{ij})$ is odd for all $i$ and $j$. If $B_1 \in \Lambda_1$, then $B_2 \notin \Lambda_2$ by the definition of $\Lambda_i$ ($i = 1, 2$). In this case we can replace the pair $\Lambda_1, \Lambda_2$ by $\Lambda_1 - \{B_1\}, \Lambda_2 \cup \{B_2\}$, and the result follows from the proof in Case 1. Therefore, we may assume that $B_i \notin \Lambda_i$ for $i = 1, 2$. Since each $\Gamma_{i12}$ is ESP, $\Lambda_i \cup \{B_i\}$ has an equitable subpartition ($\Lambda_1^1 \cup \{B_i\}, \Lambda_2^2$). Then it is straightforward to verify that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^2)$ is an equitable subpartition of $\Lambda$.

Let $G$ be a graph. A triangle $T$ of $G$ is called stable if $G \setminus V(T)$ is connected and every vertex of $T$ has degree at least three in $G$. The 3-sum of two graphs $G_1$ and $G_2$ is obtained by
identifying a stable triangle of $G_1$ with a stable triangle of $G_2$.

**Corollary 2.5.** Let $G$ be the 3-sum of $G_1$ and $G_2$ over a triangle $x_1x_2x_3$. For $i = 1, 2$ and $1 \leq j < k \leq 3$, let $G_{ijk}$ be obtained from $G_i$ by adding a new vertex $x_{ijk}$ and two new edges $x_{ijk}x_j$ and $x_{ijk}x_k$. If all $G_{ijk}$ are ESP, then so is $G$.

### 3 A decomposition of CM graphs

Let a $\Delta$-graph be obtained from a triangle $xyz$ by adding three internally vertex disjoint paths, one from $x$ to $y$, one from $y$ to $z$, and one from $z$ to $x$. Notice that a $\Delta$-graph is a special odd ring. A rooted graph consists of a graph $G$ and a specified set $F$ of edges such that each $f \in F$ belongs to a triangle and each triangle in $G$ contains at most one edge from $F$. By adding pendent triangles to the rooted graph $G$ we mean the following operation: to each edge $f = xy$ in $F$, we introduce a new vertex $z_f$ and two new edges $xz_f$ and $yz_f$. The following is the main result of this section.

**Theorem 3.1.** For any graph $G$, at least one of the following holds.

(i) $G$ is the $k$-sum of two smaller graphs, for some $k = 0, 1, 2, 3$;
(ii) $G$ has a $\Theta$-graph, a $W$-graph, or a $\Delta$-graph;
(iii) $G$ is obtained from a rooted 2-connected line graph by adding pendent triangles.

We break the proof of this result into several lemmas. A path with end-vertices $x$ and $y$ is called an $xy$-path.

**Lemma 3.1.** Let $H$ be a subdivision of $K_4$ and let $a$ and $b$ be two of the four degree-three vertices. Let $G$ be obtained from $H$ by adding edges such that all these edges are incident with either $a$ or $b$. Then $G$ has a $W$-graph.

**Proof.** Let $c$ and $d$ be the other two vertices of $H$ of degree three. For distinct vertices $x, y$ in $\{a, b, c, d\}$, we denote by $P_{xy}$ the path obtained by subdividing the edge $xy$ of $K_4$. Without loss of generality, we assume that all these paths are induced. Therefore, edges not in any of these paths must be between $\{a, b\}$ and $V(P_{cd}) - \{c, d\}$. If all these edges are incident with only one of $a$ and $b$, then it is easy to see that $G$ is a $W$-graph. Thus we may assume that both $a$ and $b$ have neighbors in $V(P_{cd}) - \{c, d\}$. For each vertex $x$ in $V(P_{cd})$, let $P_{cx}$ be the unique $cx$-path of $P_{cd}$. We choose $x$ in $V(P_{cd})$, with $V(P_{cx})$ minimal, such that $V(P_{cx} - \{c\})$ contains both neighbors of $a$ and $b$. Then it is straightforward to verify that $G\backslash(V(P_{cd}) - V(P_{cx}))$ is a $W$-graph.

**Lemma 3.2.** Let $G$ be a graph with at least six vertices and let $xy$ be an edge of $G$ such that $G\backslash\{x, y\}$ is disconnected. Then $G$ is the 2-sum of two smaller graphs over $xy$, unless $G\backslash\{x, y\}$
has only two components with one being a single vertex.

Proof. If all components of $G \setminus \{x, y\}$ are single vertices, let $G'_1$ consist of two of these vertices. If $G \setminus \{x, y\}$ has a component with two or more vertices, let $G'_1$ be such a component. Let $G'_2 = G \setminus (V(G'_1) \cup \{x, y\})$. Clearly, each $G'_i$ has at least two vertices, unless $G \setminus \{x, y\}$ has only two components with one being a single vertex. For $i = 1, 2$, let $G_i$ be obtained from $G \setminus V(G'_i)$ by adding a new vertex $z_i$ and two new edges $z_ix$ and $z_iy$. Then both $G_1$ and $G_2$ have fewer vertices than $G$ and it is straightforward to verify that $G$ is the 2-sum of $G_1$ and $G_2$. ■

A diamond is a graph obtained from $K_4$ by deleting an edge. The following is a corollary of the last two lemmas.

Lemma 3.3. If a graph $G$ has a diamond $D$, then at least one of the following holds.
(i) $D$ has a vertex of degree two in $G$;
(ii) $G$ can be expressed as the 2-sum of two smaller graphs, and the two triangles of $D$ are contained in different parts;
(iii) $G$ has a $W$-graph.

Proof. Let $V(D) = \{a, b, c, d\}$ and let $a$ and $b$ have degree three in $D$. If $c$ and $d$ are contained in the same component of $G \setminus \{a, b\}$, then $G \setminus \{a, b\}$ has an induced $cd$-path $P$. By applying Lemma 3.1 to $D \cup P$ we deduce that (iii) holds. Therefore, we may assume that $c$ and $d$ are contained in different components $G_c$ and $G_d$ of $G \setminus \{a, b\}$. It is clear that (i) holds if $G_c$ or $G_d$ consists of only one vertex. On the other hand, if both $G_c$ and $G_d$ contain two or more vertices, then we deduce from the proof of Lemma 3.2 that (ii) holds.

An edge $e = xy$ is called a chord of a cycle $C$ if $e \notin E(C)$ yet both $x$ and $y$ are in $V(C)$. A $\Theta_1$-graph is obtained from a cycle of length at least six by adding precisely one chord such that no triangle is created. A $\Theta_2$-graph is obtained from a cycle of length at least six by adding precisely two chords $xy$ and $xz$ such that $yz$ is an edge of the cycle; we shall call $xyz$ the inscribed triangle of the $\Theta_2$-graph.

Lemma 3.4. If a graph $G$ has a $\Theta_1$-graph $H$ with chord $e$, then at least one of the following holds.
(i) $G$ has a $\Theta_2$-graph whose inscribed triangle contains $e$;
(ii) $G$ can be expressed as the 2-sum of two smaller graphs over $e$;
(iii) $G$ has a $W$-graph.

Proof. Let $a, b \in V(H)$ be the two ends of $e$ and let $P_1$ and $P_2$ be the two components of $H \setminus \{a, b\}$. If $P_1$ and $P_2$ are contained in different components of $G \setminus \{a, b\}$, then we deduce from Lemma 3.2 that (ii) holds. Next we consider the case when $G \setminus \{a, b\}$ has a component that contains both paths $P_1$ and $P_2$. In this component, we choose a shortest path $P$ between
$P_1$ and $P_2$. Then $P$ is an induced path. Let $x_0, x_1, \ldots, x_p, x_{p+1}$ be the vertices of $P$ such that $x_0 \in V(P_1)$, $x_{p+1} \in V(P_2)$, and they are ordered as in $P$. From the minimality of $P$, no $x_i$ $(i > 1)$ has a neighbor in $P_1$ and no $x_i$ $(i < p)$ has a neighbor in $P_2$. Let us now distinguish among three cases.

**Case 1.** $x_1$ has three or more neighbors in $P_1$. In this case, $V(P_1) \cup \{a, b, x_1\}$ induces a $W$-graph and thus (iii) holds. By symmetry, (iii) also holds if $x_p$ has three or more neighbors in $P_2$.

**Case 2.** $x_1$ has precisely one neighbor in $P_1$ and $x_p$ has precisely one neighbor in $P_2$. In this case, we deduce from Lemma 3.1 that (iii) holds.

**Case 3.** If none of the previous cases happens, then, by symmetry, we may assume that $x_1$ has precisely two neighbors in $P_1$. It is easy to see that (iii) holds if no $x_i$ $(1 \leq i \leq p)$ is adjacent to any of $a$ and $b$. So we can choose the smallest $i$ in $\{1, 2, \ldots, p\}$ such that $x_i$ is adjacent to $a$ or $b$, say $a$. If $x_i$ is not adjacent to $b$, then (iii) holds since $V(P_1) \cup \{a, b, x_1, \ldots, x_i\}$ induces a subdivision of $K_4$. Thus we can assume that $x_i$ is also adjacent to $b$. If $i$ is 1 or $p$, then it is not difficult to see that (iii) holds again. If $1 < i < p$, then (ii) holds since the subgraph induced by $V(P'_1) \cup V(P_2) \cup \{a, b, x_1, \ldots, x_i\}$ is a $\Theta_2$-graph with inscribed triangle $abx_i$, where $P'_1$ is the part of $P_1$ from $x_0$ to $\{a, b\}$ that avoids the other neighbor of $x_1$ in $P_1$.

If $T$ is a triangle of a graph $G$ for which $G \setminus V(T)$ has more components than $G$, then $T$ is called a separating triangle.

**Lemma 3.5.** If a graph $G$ has a $\Theta_2$-graph $H$ with inscribed triangle $xyz$, then either $G$ has a $W$-graph, or $xyz$ is a separating triangle of $G$ such that $H \setminus \{x, y, z\}$ is not entirely contained in any component of $G \setminus \{x, y, z\}$.

**Proof.** Rename the vertices if necessary, we may assume $xy$ and $xz$ are two chords of $H$. Let $P_1$, $P_2$ be the two path of $H \setminus \{x, y, z\}$ such that $y$ is adjacent to an end of $P_1$. Suppose $H \setminus \{x, y, z\}$ is contained in a component of $G \setminus \{x, y, z\}$. Then $G \setminus \{x, y, z\}$ has a path $P$ between $P_1$ and $P_2$. Let us choose $P$ as short as possible. It follows that $P$ is an induced path. Let $x_0, x_1, \ldots, x_p, x_{p+1}$ be the vertices of $P$ such that $x_0 \in V(P_1)$, $x_{p+1} \in V(P_2)$, and they are ordered as in $P$. From the minimality of $P$, no $x_i$ $(i > 1)$ has a neighbor in $P_1$ and no $x_i$ $(i < p)$ has a neighbor in $P_2$. We now prove that $G$ has a $W$-graph.

**Case 1.** $x_1$ has three or more neighbors in $P_1$. In this case, $V(P_1) \cup \{x, y, x_1\}$ induces a $W$-graph. By symmetry, $G$ also has a $W$-graph if $x_p$ has three or more neighbors in $P_2$.

**Case 2.** $x_1$ has precisely one neighbor in $P_1$ and $x_p$ has precisely one neighbor in $P_2$. If no $x_i$ $(1 \leq i \leq p)$ is adjacent to any of $y$ and $z$, then $V(H) \cup V(P)$ induces a $W$-graph. On the other hand, if some $x_i$ is adjacent to $y$ or $z$, say $z$, then, by applying Lemma 3.1 to the subgraph induced by $V(P_1) \cup \{x, y, z, x_1, \ldots, x_i\}$, where $i \geq 1$ is minimum for which $x_iz$ is an edge, we deduce that $G$ has a $W$-graph.

**Case 3.** If none of the previous cases occurs, then, by symmetry, we may assume that $x_1$
has precisely two neighbors in $P_1$. We may also assume that each $x_i$ is adjacent to at most one of $x$ and $y$, since otherwise $\{x,y,z,x_i\}$ induces either a $K_4$, which is a $W$-graph, or a diamond, in which cases we deduce from Lemma 3.3 that $G$ has a $W$-graph. It follows that $V(P) \cup V(P_2) \cup \{x\}$ contains an induced path $P'$ from $x_1$ to $\{x, y\}$. Therefore, $V(P_1) \cup V(P') \cup \{x, y\}$ induces a subdivision of $K_1$ and the proof is complete.

To state the next lemma, we need to define several more graphs. $F_5$ is obtained from a path on five vertices by adding a new vertex and making it adjacent to all vertices in the path. $F_5^+$ is obtained from $F_5$ by adding an edge between the two nonadjacent vertices of degree three. A $\Theta_3$-graph is obtained from $K_{2,3}$ by subdividing an edge an arbitrary number of times and then adding an edge between the two vertices of degree three. A $K_4^+$-graph is obtained from $K_4$ by subdividing at least two of the three edges from some star. A $K_4^+\gamma$-graph is obtained from a $K_4$ with vertex set $\{a, b, c, d\}$ by subdividing $ab$ at least once and then adding new vertex $x$ and two new edges $xa$ and $xc$. We shall call an induced $K_{1,3}$ a claw.

**Lemma 3.6.** A 2-connected graph has a claw if and only if it has an induced subgraph that is isomorphic to $F_5$, $F_5^+$, a $\Theta$-graph, a $\Theta_i$-graph $(i = 1, 2, 3)$, a $K_4^+$-graph, or a $K_4^+\gamma$-graph.

**Proof.** The “if” part is obvious since all the listed graphs have claws. To prove the “only if” part, let $G$ be a 2-connected graph with a claw. Clearly, we may assume that $G$ is minimal with this property, that is, every proper induced subgraph of $G$ is either not 2-connected or claw-free. In particular, for every vertex $z$ of $G$, every block of $G \setminus z$ is claw-free. Let edges $xa_1, xa_2,$ and $xa_3$ form a claw. Then we deduce that, for each vertex $z \notin \{x, a_1, a_2, a_3\}$, $x$ is a cut-vertex of $G \setminus z$ that separates some $a_i$ from some other $a_j$. Equivalently, for every vertex $z \notin \{x, a_1, a_2, a_3\}$, the set $\{x, z\}$ is a vertex-cut of $G$ that separates some $a_i$ from some other $a_j$.

Since $G$ is 2-connected and $x$ has degree at least three, there must exist a vertex $y$ other than $x$ such that the degree of $y$ is at least 3. Now the 2-connectivity of $G$ guarantees the existence of a path in $G$ from $a_i$ to $y$ which avoids $x$; by taking an appropriate section of this path, we see that for $i = 1, 2, 3$, there is a vertex $b_i \neq x$ and a path $P_i$ from $x$ to $b_i$ such that $a_i$ is in $P_i$, $b_i$ has degree at least three, and all interior vertices of $P_i$ have degree two in $G$ (possibly $b_i = a_i$). We claim that $V(P_1 \cup P_2 \cup P_3) = V(G)$. Suppose, on the contrary, that some vertex $z$ of $G$ is not in $V(P_1 \cup P_2 \cup P_3)$. Then, without loss of generality, we may assume that $a_1$ and $a_2$ are separated from $a_3$ by $\{x, z\}$. Let $G_3$ be the component of $G \setminus \{x, z\}$ that contains $a_3$. Then $b_3$ is also contained in $G_3$. Let $G'_3$ be the subgraph of $G$ induced by $V(G_3) \cup \{x, z\}$. Since $G$ is 2-connected, $G'_3$ must have an $xz$-path $P$ with at least one interior vertex. Choose such a path $P$ as short as possible. Then $P$ is an induced path, except for a possible edge $xz$. It follows that $V(G'_3) - V(P) \neq \emptyset$ since all vertices in $G'_3 \setminus (V(G_3) - V(P))$ have degree at most two while $b_3$ is a vertex in $G'_3$ of degree at least three. Therefore, $G \setminus (V(G'_3) - V(P))$ is a proper induced subgraph and thus it should be either claw-free or not 2-connected. However, this graph has a claw $\{xa_1, xa_2, xa\}$, where $a$ is the neighbor of $x$ in $P$, and it is also 2-connected since it is
obtained from a 2-connected graph $G$ by replacing a part of a 2-separation with a path. This contradiction completes the proof of our claim.

Depending on the relationship between $b_i$‘s, we distinguish among the following three cases.

**Case 1.** $b_1 = b_2 = b_3$. In this case, there are two subcases. If $xb_1$ is not an edge, then $G$ is a $\Theta$-graph. If $xb_1$ is an edge, then $G$ is either a $\Theta_1$-graph or a $\Theta_3$-graph.

**Case 2.** $b_1 = b_2 \neq b_3$. In this case, the above claim implies that $b_3$ must be adjacent to $b_1$ and $x$. It follows that $b_i = a_i$ for some $i = 1, 2$, for otherwise, $G$ has a subdivision of $K_{2,3}$ as a proper induced subgraph, which is impossible. Therefore, $G$ is a $\Theta_2$-graph.

**Case 3.** $b_1, b_2$ and $b_3$ are all distinct. In this case, by the above claim each $b_i$ is adjacent to at least one other $b_j$. It follows that there are at least two edges between $b_1, b_2$ and $b_3$. We first consider the case when some two of $b_1, b_2$ and $b_3$, say $b_1$ and $b_3$, are not adjacent. Since each $b_i$ has degree at least three, for $i = 1, 3$, $xb_i$ must be an edge not in $P_i$. From the minimality of $G$ we deduce that, for $i = 1, 3$, $a_i$ is the only interior vertex of $P_i$, and in addition, $a_2 = b_2$. Thus $G = F_5$. Next, we assume that $b_1b_2b_3$ is a triangle. If there are no other edges, then $G$ is a $K^+_4$-graph. Thus we may assume that $xb_1$ is an edge not in $P_i$ for some $i$, say $i = 1$. From the minimality of $G$ it is not difficult to see that $a_i = b_i$ for some $i$, say $i = 2$. Then $a_3 \neq b_3$, as $b_2b_3$ is an edge while $a_2a_3$ is not. Also from minimality of $G$ we deduce that $a_1$ is the only interior vertex of $P_1$. Now it is straightforward to verify that $G$ is either $F^+_5$ when $xb_3$ is an edge, or a $K^+_4$-graph when $xb_3$ is not an edge. ■

**Lemma 3.7.** Let $T$ be a separating triangle of a 2-connected graph $G$. Then at least one of the following holds.

(i) $G$ is the $k$-sum of two smaller graphs, for some $k = 2, 3$;

(ii) $G$ has a $\Delta$-graph or a $W$-graph;

(iii) $G \setminus V(T)$ has precisely two components, one of which is a single vertex $x$ of degree two.

**Proof.** Let $V(T) = \{x_1, x_2, x_3\}$ and let us assume that (i) does not hold. We need to show that either (ii) or (iii) holds. We first consider the case when $G \setminus V(T)$ has exactly two components $G_1$ and $G_2$. Since $G$ is not the 3-sum of two other graphs, for some $x_i$ and $G_j$, say $i = j = 1$, $x_1$ has no neighbors in $G_1$. It follows that $G_1$ is a component of $G \setminus \{x_2, x_3\}$. But $G$ is not the 2-sum of two smaller graphs, we conclude from Lemma 3.2 that $G_1$ is a single vertex of degree two and thus (iii) holds. Next, we consider the case when $G \setminus V(T)$ has more than two components. For each component $H$ of $G \setminus V(T)$, let $T(H)$ be the set of vertices in $T$ that have neighbors in $H$. Let $G_1, G_2$, and $G_3$ be components of $G \setminus V(T)$ such that $|T(G_1)| \leq |T(G_2)| \leq |T(G_3)|$. From Lemma 3.2 we deduce that, for each $i \neq j$, $G \setminus \{x_i, x_j\}$ has at most two components. It follows that, by renaming the vertices of $T$ if necessary, we have $T(G_1) \supseteq \{x_2, x_3\}$, $T(G_2) \supseteq \{x_1, x_3\}$, and $T(G_3) \supseteq \{x_1, x_2\}$. Let $i, j, k$ be a permutation of 1, 2, 3. It is clear that we can find an $x_ix_k$-path $P_{jk}$ such that the path has at least one interior vertex and all its interior vertices are in $G_i$. Let us choose such $P_{jk}$ as short as possible. Then $V(P_{jk})$ induces a cycle. Now it is easy.
to see that (ii) holds since either the subgraph induced by $V(P_{12} \cup P_{23} \cup P_{31})$ is a $\Delta$-graph or some $x_i$, say $x_1$, has a neighbor in $P_{23}\{x_2,x_3\}$, which implies that the subgraph induced by $V(P_{23}) \cup \{x_1\}$ is a $W$-graph.

**Lemma 3.8.** Let $X$ be a set of degree-two vertices in a 2-connected graph $G$ such that each vertex in $X$ is in a triangle. If $G \setminus X$ is a 2- or 3-sum of two smaller graphs, then either $G$ is a 2- or 3-sum of two smaller graph, or $G$ has a $\Delta$- or $W$-graph.

**Proof.** If $X$ contains two adjacent vertices, then it is easy to see that $G = K_3$ and thus the result holds trivially. Therefore, we may assume that no two vertices in $X$ are adjacent. Suppose $G \setminus X$ is the 2-sum of two smaller graphs $G'_1$ and $G'_2$. It is clear that $X$ can be partitioned into $X_1$ and $X_2$ such that for each $x$ in $X_i$ (i = 1, 2), the two neighbors of $x$ are both in $G'_i$. For $i = 1, 2$, let $G_i$ be obtained from $G'_i$ by putting the vertices in $X_i$ back. Then it is easy to verify that $G$ is the 2-sum of $G_1$ and $G_2$, both are smaller than $G$. Next, suppose $G' \setminus X$ is a 3-sum of two smaller graphs over a triangle $T$. If each $x$ in $X$ has at most one neighbor in $T$, then, similar to the previous case, $G$ is a 3-sum of two smaller graphs. If some $x$ in $X$ has both neighbors in $T$, then $G \setminus V(T)$ has three or more components. It follows from Lemma 3.7 that either $G$ is a 2- or 3-sum of two smaller graphs, or $G$ has a $\Delta$- or $W$-graph.

We also need the following characterization of line graphs [1].

**Beineke’s Theorem.** A graph is a line graph if and only if it does not have any of the nine graphs below as an induced subgraph.

![Figure 2. The nine forbidden induced subgraphs.](image)

**Proof of Theorem 3.1.** Let $G$ be a graph for which neither (i) nor (ii) holds. We need to
show that (iii) must hold. Clearly, \( G \) is 2-connected. Let us also assume that \( G \) is not a line graph.

We first consider the case when \( G \) has at most five vertices. Since \( G \) is not a line graph and it does not have \( W \)-graphs, we conclude from Beineke’s Theorem that \( G \) has a claw. Then, since \( G \) has no \( \Theta \)-graphs, we deduce from Lemma 3.6 that \( G \) is the graph obtained from \( K_{2,3} \) by adding an edge between the two vertices of degree three. Clearly, (iii) holds in this case.

Next, we assume that \( G \) has at least six vertices. Let \( X \) be the set of vertices \( x \) for which there is a separating triangle \( T_x \) such that \( x \) is a component of \( G \setminus V(T_x) \). Since \( G \) is 2-connected and has no \( W \)-graphs, each \( x \) in \( X \) must have degree two. Then we conclude from Lemma 3.7 and the fact \(|V(G)| \geq 6\) that \( T_y \neq T_z \) whenever \( y, z \in X \) with \( y \neq z \). In addition, we prove that \( V(T_x) \cap X = \emptyset \) for all \( x \in X \). Suppose on the contrary, there exist \( x \in X \) and \( y \in V(T_x) \cap X \). Let \( z_1, z_2 \) be the other two vertices of \( T_x \). Then it is clear that \( G \setminus \{z_1, z_2\} \) has at least three components. Thus, by Lemma 3.2, \( G \) is a 2-sum of two smaller graphs, a contradiction. In conclusion, if \( u_x, v_x \) are the two neighbors of each \( x \in X \), then \( G_X = G \setminus X \) is a rooted graph with the set of root edges \( F = \{u_xv_x : x \in X\} \). Clearly, \( G_X \) is 2-connected and \( G \) is obtained from \( G_X \) by adding pendent triangles.

It remains to prove that \( G_X \) is a line graph. Suppose it is not. We first observe from Beineke’s Theorem and Lemma 3.6 that \( G_X \) has at least five vertices. Then we claim that, in \( G_X \), every vertex in a diamond must have degree greater than two. Suppose \( u \) has degree two and is in a diamond (one of its triangles is \( uvw \)). Since \( u \) is not included in \( X \), it must have degree three or more in \( G \) and thus must have a neighbor \( x \) in \( X \). It follows that \( T_x \) is the triangle \( uvw \). Let \( G' = G \setminus \{x \} \), the graph obtained from \( G_X \) by putting \( x \) back. Then we see that the edge \( xu \) is a component of \( G' \setminus \{v, w\} \). Therefore, from Lemma 3.2 and Lemma 3.8 we conclude that either (i) or (ii) holds for \( G \), a contradiction and thus the claim is proved. It follows from this claim, Lemma 3.3, and Lemma 3.8 that \( G_X \) has no diamonds.

Since \( G_X \) is not a line graph and it has no diamonds, by Beineke’s Theorem, \( G_X \) has a claw. Since \( G_X \) has no \( \Theta \)-graphs and \( W \)-graphs either, we deduce from Lemma 3.6 that \( G_X \) has a \( \Theta_i \)-graph for \( i = 1 \) or \( 2 \). Then, by Lemma 3.4 and Lemma 3.8 we deduce that \( i = 2 \). Let \( T \) be the inscribed triangle of this \( \Theta_2 \)-graph. By Lemma 3.5, Lemma 3.7, and Lemma 3.8, we deduce that \( G_X \setminus T \) has a component which is a vertex of degree two; this vertex together with \( T \) induce a diamond in \( G_X \), a contradiction.

4 A proof of Theorem 1.1.

Let \( \mathcal{L} \) be the class of graphs that do not have \( \Theta \)-graphs, \( W \)-graphs, and odd rings. The next is the main result of this section, which clearly includes Theorem 1.1.
Theorem 4.1. The following are equivalent for a graph $G$.

(i) $G$ is CM;
(ii) $G$ is ESP;
(iii) $G$ is in $\mathcal{L}$.

Again, we prove the theorem by proving a sequence of lemmas.

Lemma 4.1. If $G$ is a $\Theta$-graph, a $W$-graph, or an odd ring, then $G$ is not a CM graph.

Proof. Let $G$ be a $\Theta$-graph. To justify the statement, we consider the following weight function $w$ on $V(G)$: $w(v) = 1$ if $v$ is of degree two and 2 otherwise. Then it is a routine matter to check that the size of a maximum cycle packing in $G$ is 1, while the size of a minimum feedback set is 2. Hence, by definition, $G$ is not a CM graph.

Let $G$ be a $W$-graph with weight $w(v) = 1$ on each vertex $v$. Then it is easy to see that the size of a maximum cycle packing in $G$ is 1, while the size of a minimum feedback set is 2. So $G$ is not a CM graph according to the definition.

Let $G$ be an odd ring with weight $w(v) = 1$ on each vertex $v$ and let $M$ be the cycle-vertex incidence matrix of $G$. In view of the equivalent definition of Mengerian hypergraph in terms of a TDI system, to verify the statement, we may turn to prove that at least one of

$$\begin{align*}
\max & \quad y^T e \\
\text{s.t.} & \quad y^T M \leq e^T \\
\min & \quad e^T x \\
\text{s.t.} & \quad Mx \geq e \\
y & \geq 0 \\
x & \geq 0
\end{align*}$$

has no integral optimal solution. To this end, it suffices to show that the optimal objective value of the above linear programs is not integral.

Recall that $G$ is a graph obtained from an odd cycle by replacing each edge $e = uv$ with either a cycle $C_e$ containing $e$ or two triangles $uab$, $vcd$ together with two vertex-disjoint paths $P_{ab}$ and $P_{cd}$ between $\{a, b\}$ and $\{c, d\}$. In the former case, define $x(u) = x(v) = 1/2$ and $y(C_e) = 1/2$; in the latter case, define $x(u) = x(a) = x(c) = x(v) = 1/2$ and $y(C) = 1/2$ for each $C$ of the following three cycles: $uab$, $vedv$ and $abP_{ab}dcP_{ac}$. For all the remaining vertices $v$ and all the remaining cycles $C$, we define $x(v) = 0$ and $y(C) = 0$. Clearly, $x$ and $y$ are both well defined. Moreover, $y$ is a feasible solution to the primal program and $x$ is a feasible solution to the dual. Now let $t$ denote the number of vertices $v$ with $x(v) = 1/2$. Then the construction of $G$ implies that $t$ is odd. Since $y^T e = e^T x = t/2$, by the duality theory of linear programming [7] $y$ and $x$ are optimal solutions to the above programs, respectively, and hence neither of the programs has an integral optimal solution, completing the proof. $\blacksquare$

Lemma 4.2. If $G \in \mathcal{L}$ is the 0- or 1-sum of $G_1$ and $G_2$, then both $G_1$ and $G_2$ are in $\mathcal{L}$.

Proof. This is clear from the definition of $\mathcal{L}$ since both $G_1$ and $G_2$ are induced subgraphs
of $G$.

**Lemma 4.3.** If a 2-connected graph $G \in \mathcal{L}$ is the 2-sum of two smaller graphs $G_1$ and $G_2$, then both $G_1$ and $G_2$ are in $\mathcal{L}$.

**Proof.** Suppose some $G_i$, say $G_1$, has an induced subgraph $H$ which is a $\Theta$-graph, a $W$-graph, or an odd ring. We need to show that $G$ has a $\Theta$-graph, a $W$-graph, or an odd ring. Let $x, y$ be the common vertices of $G_1$ and $G_2$, and let $z_1$ be the only vertex in $G_1 \setminus V(G)$. If $z_1 \not\in V(H)$, then we are done since $H$ is an induced subgraph of $G$. If $z_1 \in V(H)$, then both $x$ and $y$ are in $H$ since $H$ has minimum degree at least two. Notice that $\Theta$-graphs do not have triangles and triangles in $W$-graphs only contain vertices of degree greater than two. Thus $H$ can only be an odd ring. It follows from the 2-connectivity of $G$ that $G_2$, and hence $G_2 \setminus z_2$, is also 2-connected. Therefore, $xy$ is contained in an induced cycle $C$ of $G_2 \setminus z_2$. Now it is clear that $V(H \setminus z_1) \cup V(C)$ induces an odd ring in $G$, as required.

**Lemma 4.4.** Let $G \in \mathcal{L}$ be the 3-sum of $G_1$ and $G_2$ over a triangle $x_1x_2x_3$. For $i = 1, 2$, and $1 \leq j < k \leq 3$, let $G_{ijk}$ be obtained from $G_i$ by adding a new vertex $x_{ijk}$ and two new edges $x_{ijk}x_j$ and $x_{ijk}x_k$. Then all $G_{ijk}$ are in $\mathcal{L}$.

**Proof.** Suppose to the contrary that some $G_{ijk}$, say $G_{112}$, has an induced subgraph $H$ which is a $\Theta$-graph, a $W$-graph, or an odd ring. We aim to show that $G$ has a $\Theta$-graph, a $W$-graph, or an odd ring. Like in the proof of the last lemma, we may assume that $x_{112}$ is in $H$ and $H$ is an odd ring which contains the entire triangle $x_{112}x_1x_2$. As $x_1x_2x_3$ is a stable triangle in $G_2$, by definition there is a path, other than $x_1x_2$, in $G_2$ from $x_1$ to $x_2$ which avoids $x_3$; let $P$ be such a path with the minimum length. Then $P$ is an induced path in $G_2$. Observe that $x_3$ is adjacent to no vertex in $P - \{x_1, x_2\}$, for otherwise $P \cup \{x\}$ induces a $W$-graph in $G$, a contradiction. Hence no vertex in $P - \{x_1, x_2\}$ is adjacent to any vertex in $H - \{x_{112}, x_1, x_2\}$ for $G$ is the 3-sum of $G_1$ and $G_2$ over $x_1x_2x_3$. It follows that the graph obtained from $H$ by replacing the path $x_1x_{112}x_2$ with $P$ is an odd ring of $G$, a contradiction.

**Lemma 4.5.** Let $G'$ be the graph obtained from a graph $G$ by subdividing an edge $yz$ with a vertex $x$. If $yz$ is contained in no triangle of $G$, then $G$ is ESP provided $G'$ is ESP.

**Proof.** We first make the natural correspondence between cycles in $G'$ and $G$ more precise. For each cycle $C$ in $G'$, let $\phi(C)$ be the cycle in $G$ such that, if $x$ is not in $C$ then $\phi(C) = C$, and if $x$ is in $C$ then $\phi(C)$ is obtained from $C \setminus x$ by adding the new edge $yz$. It is not difficult to see that $\phi$ is a 1-1 mapping. Let $\mathcal{C}$ be a set of cycles in $G$. Define $\mathcal{C}' = \{\phi^{-1}(C) : C \in \mathcal{C}\}$. Since $G'$ is ESP, $\mathcal{C}'$ has an equitable subpartition $(\mathcal{C}_1', \mathcal{C}_2')$. Now, for $i = 1, 2$, let $\mathcal{C}_i = \{\phi(C) : C \in \mathcal{C}_i'\}$. Then it is straightforward to verify that $(\mathcal{C}_1, \mathcal{C}_2)$ is an equitable subpartition of $\mathcal{C}$.

Two edges in a graph are in *series* if they form a minimal edge cut. It is not difficult to see
that being in series is an equivalence relation. We call each equivalence class a *series family*. A series family is *trivial* if it has only one edge. A graph is *subcubic* if it has maximum degree at most three. If a vertex $x$ has degree three, then the subgraph formed by the three edges incident with $x$ is called a *triad* with center $x$. We shall follow convention and let $L(H)$ stand for the line graph of a graph $H$.

**Lemma 4.6.** Let $G$ be obtained from a rooted 2-connected line graph $L(H)$ by adding pendent triangles. Suppose $H$ is subcubic and none of its cycles has chords. Then $G$ is ESP if it has no odd rings.

**Proof.** Clearly, we may assume that $H$ has no isolated vertices. Let us make some further observations about $H$.

(1) $H$ is connected and its only cut edges are the pendent edges.

This follows from our assumption that $L(H)$ is 2-connected.

(2) Every non-pendent edge of $H$ is contained in a nontrivial series family of $H$.

Assume the contrary: there exists a non-pendent edge $e = xy$ for which $\{e\}$ is a series family. It follows that $H \setminus e$ has no cut edge that separates $x$ from $y$. By (1), $H \setminus e$ is connected and thus $H \setminus e$ has two edge-disjoint $xy$-paths. In fact, these two paths are internally vertex-disjoint because $H$ is subcubic. Thus $e$ is a chord of the cycle formed by these two paths, a contradiction.

(3) If $F$ is a nontrivial series family of $H$ with $|F| = k$ odd, then $F$ has two incident edges $xy$ and $xz$ such that they are the only two edges of $H$ that are incident with $x$.

To prove (3), notice that $G \setminus F$ has exactly $k$ components. These components can be cyclically ordered, say $G_1, G_2, \ldots, G_k$, such that, for each $i$, there is an edge $e_i = x_iy_i$ of $F$ with $x_i \in V(G_i)$ and $y_i \in V(G_{i+1})$, where the subscript is taken modulo $k$. Let $I_i$, where $t = 1, 2$, be the set of indices $i$ for which $G_i$ has $t$ vertices, and let $I_3$ be the remaining indices. For each $i \in I_2$, it is clear that the only edge $f_i$ of $G_i$ shares a common end with $e_{i-1}$ and $e_i$. For each $i \in I_3$, it can be seen from (1) that $y_{i-1} \neq x_i$. From the definition of a series family, we deduce that $G_i$ has no cut edge that separates $y_{i-1}$ from $x_i$. Hence $G_i$ has a cycle $C_i$ which contains $y_{i-1}$ and $x_i$ in $G_i$ (recall the proof of (2)). Assume $I_1 = \emptyset$. Let $R$ be the subgraph of $H$ induced by the union of $\{e_i : 1 \leq i \leq k\}$, $\{f_i : i \in I_2\}$, and $E(C_i)$, for all $i \in I_3$. Since no cycle of $H$ contains chords, each of the two sections of $C_i$, for any $i \in I_3$, between $y_{i-1}$ and $x_i$ has length at least two. Now it is not difficult to check that $L(R)$ is an odd ring, which is a contradiction. It follows that $I_1 \neq \emptyset$ and thus (3) is proved.

(4) We may assume that $H$ has no triangles.

Let $K_4 \setminus e$ be obtained from $K_4$ by deleting an edge, $K_{2,3}^+$ be obtained from $K_{2,3}$ by adding an edge between the two vertices of degree three, and $F_4$ be obtained from a path on four vertices by adding a new vertex of degree four. Suppose $T$ is a triangle of $H$. Since no cycle of $H$ has a
chord, the only paths between any two vertices of $T$ are the two in $T$. It follows from (1) and (3) that $H$ is obtained from $T$ by adding at most two pendent edges. Thus $L(H)$ can only be $K_3, K_4\setminus e$ or $F_4$. Since $G$ contains no odd ring, it is straightforward to verify that either $G$ is in \{ $K_3, K_4\setminus e, F_4, K_{2,3}^+$ \} or $G$ can be constructed from graphs in \{ $K_3, K_4\setminus e, F_4, K_{2,3}^+$ \} by 2-sums. Observe that all graphs in \{ $K_3, K_4\setminus e, F_4, K_{2,3}^+$ \} are ESP, so (4) follows from Corollary 2.4.

(5) We may assume that each nontrivial series family of $H$ contains an even number of edges.

For each nontrivial series family $F$ with $|F|$ odd, let $e_1^F$ and $e_2^F$ be two edges as described in (3) and let $H'$ be obtained from $H$ by subdividing each $e_i^F$ exactly once. Clearly, all series family of $H'$ are even and $L(H')$ can be considered as obtained from $L(H)$ by subdividing each edge $e^F = e_1^F e_2^F$ exactly once. Since, by (4), $H$ has no triangle, $e^F$ is contained in no triangle in $G$. Thus $G'$, the graph obtained from $G$ by subdividing each $e^F$ exactly once, can also be obtained from $L(H')$ by adding pendent triangles (in the same way as getting $G$ from $L(H)$). Since subdividing edges does not introduce odd rings, $G'$ has no odd rings. In addition, by Lemma 4.5, $G$ is ESP if $G'$ is. Therefore, we can replace $G$ and $H$ by $G'$ and $H'$, respectively, and so (5) is proved.

Now let $\mathcal{C}$ be an arbitrary set of cycles of $G$. We prove in the following that $\mathcal{C}$ admits an equitable subpartition. We prove by induction on $k = |\mathcal{C}|$. The result is obvious if $k = 1$, so we assume that $k > 1$. Without loss of generality, we also assume that all cycles in $\mathcal{C}$ are chordless. Then we observe that there are three types of cycles in $\mathcal{C}$. The first type are cycles of length four or more which correspond to cycles of $H$, the second type are triangles of $L(H)$ which correspond to triads of $H$, and the third type are pendent triangles which correspond to pairs of edges $xy, xz$ of $H$ for which $x$, called the center of the pair, has degree three in $H$. Let $\mathcal{D}_1$ be the set of cycles $C$ of $H$ for which $L(C) \in \mathcal{C}$, let $\mathcal{D}_2$ be the set of triads $T$ of $H$ for which $L(T) \in \mathcal{C}$, and let $\mathcal{D}_3$ be the set of pairs $P = \{e, f\}$ of edges of $H$ for which there is a vertex $p$ of $G$ of degree two such that $P^* = \{p, e, f\}$ induces a pendent triangle of $\mathcal{C}$.

(6) For each $P = \{e_1, e_2\} \in \mathcal{D}_3$, we may assume that the triad $T = \{e_1, e_2, e_3\} \notin \mathcal{D}_2$.

Suppose we have both $P \in \mathcal{D}_3$ and $T \in \mathcal{D}_2$. Then we apply the induction hypothesis to $\mathcal{C} - \{P^*, L(T)\}$, which implies the existence of an equitable subpartition $(C_1, C_2)$ of $\mathcal{C} - \{P^*, L(T)\}$. Without loss of generality, let us assume that $d_{C_1}(e_3) \leq d_{C_2}(e_3)$. Then it is easy to verify that $(C_1 \cup \{L(T)\}, C_2 \cup \{P^*\})$ is an equitable subpartition of $\mathcal{C}$.

(7) We may assume that cycles in $\mathcal{D}_1$ are pairwise vertex-disjoint.

Suppose some $C_1$ and $C_2$ in $\mathcal{D}_1$ have a vertex in common. Then they must have an edge in common, as $H$ is subcubic. Since $C_1 \neq C_2$, we can find a maximal common path $R$ of these two cycles. Let $x_1, x_2$ be the ends of $R$ and let $T_i$ be the triad with center $x_i$ ($i = 1, 2$). Let $\mathcal{C}' = (\mathcal{C} - \{L(C_1), L(C_2)\}) \cup \{L(T_1), L(T_2)\}$. Then, using the fact that each $C_i$ is chordless, it is not difficult to see that $d_{\mathcal{C}'}(x) \leq d_{\mathcal{C}}(x)$ for all $x \in V(G)$. If each $L(T_i)$ appears precisely once in
$\mathcal{C}'$, then we can replace $\mathcal{C}$ by $\mathcal{C}'$ and (7) follows since both $L(C_1)$ and $L(C_2)$ are cycles of length at least 4 in $G$; else, let $\tilde{\mathcal{C}}$ be obtained from $\mathcal{C}'$ by removing each $L(T_i)$ with multiplicity 2, for $i = 1, 2$. Then the induction hypothesis guarantees the existence of an equitable subpartition of $\tilde{\mathcal{C}}$ by introducing the corresponding $L(T_i)$ precisely once to each part of this subpartition, we get an equitable subpartition of $\mathcal{C}$. So we are done.

(8) *Every cycle in $H$ is a disjoint union of nontrivial series families.*

To justify (8), note that for any series family $F$ and any cycle $C$ of $H$, if $C$ contains some edge in $F$, then $C$ contains all the edges in $F$. Indeed, if there exist two edges $e$ and $f$ in $F$ such that $e \in C$ while $f \notin C$, then $e$ is a cut edge of $H \setminus f$ and $C$ is a cycle in $H \setminus f$ that contains the cut edge $e$, which is impossible. Now statement (8) follows instantly from this observation and (2).

Let us contract each $C$ in $\mathcal{D}_1$ into a vertex. Then from (5), (7) and (8), it follows that the resulting graph $H'$ is bipartite. Let $X_1, X_2$ be the two color classes of $H'$. Then $\mathcal{D}_1$ is naturally partitioned into $\mathcal{D}_1^1$ and $\mathcal{D}_1^2$ such that each $\mathcal{D}_1^i$ contains those cycles in $\mathcal{D}_1$ that are contracted to a vertex in $X_i$. The partition $(X_1, X_2)$ also induces a partition $(V_1, V_2)$ of $V(H)$ such that each $V_i$ contains vertices $x$ for which either $x \in X_i$ or $x$ is in some $C \in \mathcal{D}_1^i$. Then we partition $\mathcal{C}$ into $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows. For each $C \in \mathcal{D}_1$, we put $L(C)$ in $\mathcal{C}_i$ if $C$ is not in $\mathcal{D}_1^i$. For each $T \in \mathcal{D}_2$, we put $L(T)$ in $\mathcal{C}_i$ if the center of $T$ is in $V_i$. For each $P \in \mathcal{D}_3$, we put $P^*$ in $\mathcal{C}_i$ if the center of $P$ is in $V_i$.

We prove that $(\mathcal{C}_1, \mathcal{C}_2)$ is an equitable subpartition of $\mathcal{C}$. Since $(\mathcal{C}_1, \mathcal{C}_2)$ is a partition of $\mathcal{C}$, it is clear that we only need to verify $\max\{d_{\mathcal{C}_1}(x), d_{\mathcal{C}_2}(x)\} \leq \lfloor d_{\mathcal{C}}(x)/2 \rfloor$ for all $x \in V(G)$. It follows from (6) and (7) that $d_{\mathcal{C}}(x) \leq 3$ for all $x \in V(G)$. Thus we only need to show that, if $d_{\mathcal{C}}(x) \geq 2$ then $d_{\mathcal{C}_i}(x) > 0$ for $i = 1, 2$. Observe that if a vertex of $G$ is contained in two or more cycles of $\mathcal{C}$, this vertex must be an edge $e$ of $H$. Also observe that there are only two kinds of edges in $H$: those between $V_1$ and $V_2$, and those with both ends in some $V_i$ which are precisely those in some $C \in \mathcal{D}_1$. Then the result follows from (6) and the definition of $\mathcal{C}_1$ and $\mathcal{C}_2$.

**Proof of Theorem 4.1.** The implication $(ii) \Rightarrow (i)$ is given by Corollary 2.1. The implication $(i) \Rightarrow (iii)$ follows from Corollary 2.2 and Lemma 4.1. It remains to prove the implication $(iii) \Rightarrow (ii)$: we apply induction on $|V(G)|$. The case $|V(G)| = 1$ is trivial, so we proceed to the induction step. By Lemmas 4.2–4.4 and Corollaries 2.3–2.5, we may assume that $G$ cannot be represented as the $k$-sum $(k = 0, 1, 2, 3)$ of two smaller graphs (otherwise we are done). Then we conclude from Theorem 3.1 that $G$ is obtained from a rooted 2-connected line graph $L(H)$ by adding pendent triangles. Since $G$ contains no $K_4$, $H$ is subcubic. Also notice that $H$ contains no cycle with chords, for otherwise a cycle together with a chord in $H$ would correspond to a $W$-graph (which is a subdivision of a wheel with four spokes) in $G$, a contradiction. Now we deduce from Lemma 4.6 that $G$ is ESP.
References


