UNAVOIDABLE MINORS OF GRAPHS OF LARGE TYPE

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ABSTRACT. In this paper, we study one measure of complexity of a graph, namely its type. The type of a graph \( G \) is defined to be the minimum number \( n \) such that there is a sequence of graphs \( G = G_0, G_1, \ldots, G_n \), where \( G_i \) is obtained by contracting one edge in or deleting one edge from each block of \( G_{i-1} \), and where \( G_n \) is edgeless. We show that a 3-connected graph has large type if and only if it has a minor isomorphic to a large fan. Furthermore, we show that if a graph has large type, then it has a minor isomorphic to a large fan or to a large member of one of two specified families of graphs.

1. INTRODUCTION

Graphs in this paper are finite and may have loops and multiple edges. A graph \( G \) is a minor of a graph \( H \), written \( G \leq H \), if \( G \) can be obtained from a subgraph of \( H \) by contracting edges. The following is the celebrated Robertson-Seymour Theorem [4], previously known as Wagner's Conjecture.

1.1. Theorem. Every infinite set of graphs contains two elements one of which is isomorphic to a minor of the other.

One of the central problems in matroid theory lies in determining the classes of matroids that admit an extension of the Robertson-Seymour Theorem. More precisely, it addresses the following:

1.2. Question. For which finite field \( F \) does every infinite set of matroids representable over \( F \) contain two elements one of which is isomorphic to a minor of the other?

While this problem seems very difficult, a small but significant step towards its solving has been made by one of the authors in collaboration with others [1] by proving that this theorem extends to every class of matroids that are representable over a fixed finite field and have an additional structural property: bounded type. Type \( t(M) \) of a matroid \( M \) is defined as follows: If \( E(M) = \emptyset \), then \( t(M) = 0 \). If \( E(M) \neq \emptyset \) and \( M \) is connected, then \( t(M) = 1 + \min\{t(M \setminus e), t(M/e) : e \in E(M)\} \). If \( M \) is disconnected, then \( t(M) = \max\{t(M_i)\} \), where is maximum is taken over all connected components \( M_i \) of \( M \). The mentioned result of [1] may be stated now more precisely as the following:

1.3. Theorem. Let \( F \) be a finite field, \( k \) be a nonnegative integer, and \( \mathcal{M} \) be the class of \( F \)-representable matroids of type at most \( k \). Then every infinite subset of \( \mathcal{M} \) contains two elements one of which is isomorphic to a minor of the other.
Since Theorem 1.3 answers Question 1.2 for classes of matroids of bounded type, it provides a strong motivation to characterize classes of matroids of unbounded type. Informally speaking, we would like to describe such classes by the presence of certain minors in their members. We believe that finding such descriptions for the class of all matroids is very difficult, but that it becomes easier for more restricted classes of matroids. In this paper, we give such a characterization for graphic matroids.

Even though the results of this paper have been motivated by research in matroid theory, they speak of graphs. Consequently, for the reader’s convenience, we translate the definition of type into the language of graph theory. If a graph $G$ is edgeless, then its type, $t(G)$, is zero. If $G$ has edges and $G$ is a block, then $t(G) = 1 + \min \{t(G \setminus e), t(G/e) : e \in E(G)\}$. If $G$ is not a block, then $t(G) = \max \{t(G_i)\}$, where is maximum is taken over all blocks $G_i$ of $G$. Clearly, the definitions of type for matroids and for graphs are consistent, that is, the type of a graph equals the type of the cycle matroid of the graph.

In Section 2, we examine three families of graphs: fans, multicycles, and commuticycles. For a positive integer $n$, an $n$-fan $F_n$ is the graph obtained from a path on $n$ vertices by adding a new vertex and joining it to all vertices of the path. For integers $m$ and $n$ exceeding 2 and 0, respectively, the $(m,n)$-multicycle $C_{m,n}$ is the graph obtained from a cycle on $m$ vertices by replacing each of its edges by $n$ parallel edges. An $(m,n)$-comuticycle $C^*_{m,n}$ is the planar dual of $C_{m,n}$, or, equivalently, the graph obtained from an $m$-edge bond $C^*_m$ (the graph consisting of $m$ parallel edges joining two vertices) by subdividing each of the edges by $n - 1$ new vertices. In Section 2 we show, roughly speaking, that if $m$ and $n$ are large, and so are the types of $F_n, C_{m,n}$, and $C^*_{m,n}$. We also show there that type is not monotone under the taking of minors, that is, that $G \preceq_m H$ does not imply $t(G) \leq t(H)$. We believe that this lack of monotonicity is the main reason for the difficulty in obtaining a desired characterization of classes of matroids of bounded type.

In Section 3, we characterize 3-connected graphs of large type. For a positive integer $n$, an $n$-wheel $W_n$ is the graph obtained from a cycle on $n$ vertices by adding a new vertex and joining it to all vertices of the cycle. Section 3 contains contains proof of the following two theorems.

1.4. Theorem. If $G$ is a graph that contains $F_n$ as a minor, then the type $t(G)$ of $G$ is at least $\lceil \log_2 n \rceil + 1$.

1.5. Theorem. For each positive integer $n$ exceeding 2, there is an integer $t_n$ such that if $G$ is a 3-connected graph and $t(G) \geq t_n$, then $G$ has a minor isomorphic to an $n$-spoke wheel $W_{n}$.

Note that, since, for each $n$, the fan $F_n$ is a minor of the wheel $W_n$, Theorems 1.4 and 1.5 imply the following, somewhat informal, remark, which describes the first main result of the paper.

1.6. Remark. A 3-connected graph has large type if and only if it has no minor isomorphic to large fan.

The second of the main results of the paper is a following analog of Theorem 1.5 for arbitrary graphs.

1.7. Theorem. For every integer $n$ exceeding 3, there is a number $N$ such that every graph whose type is at least $N$ has a minor isomorphic to one of $\{F_n, C_{n,n}, C^*_{n,n}\}$.
We note that, as will be demonstrated later in Section 2, graphs containing a minor isomorphic to \( C_{n,n} \) or \( C^*_{n,n} \) for a large value of \( n \) may have small type. Consequently, Theorem 1.4 has no analog for arbitrary graphs. The remainder of the paper, starting with Section 4, is devoted to proving Theorem 1.7.

The set of vertices of a graph \( G \) will be denoted by \( V(G) \), and the set of edges of \( G \) by \( E(G) \). An edge that is not a loop is a link-edge, and a non-empty maximal class of parallel link-edges is a multi-edge. If a multi-edge contains at least two edges, then the multi-edge is proper; otherwise, it is trivial. We shall write \( e \parallel uv \) to indicate that the endvertices of \( e \) are \( u \) and \( v \). If \( v \) is a vertex of a graph \( G \), then the degree of \( v \) in \( G \) is \( |E_v| + 2|L_v| \), where \( E_v \) is the set of link-edges of \( G \) incident with \( v \) and \( L_v \) is the set of loops of \( G \) incident with \( v \). If \( n \) is a nonnegative integer, then an \( n \)-path is a graph isomorphic to the path on \( n+1 \) vertices. A path of length 0 is trivial; otherwise it is proper.

If \( e \in E(G) \), then we shall use the standard notation of \( G\setminus e \) and \( G/e \) to denote the deletion of the edge \( e \) from \( G \) and the contraction of the edge \( e \) in \( G \), respectively. Also, if \( v \in V(G) \), then \( G - v \) denotes the deletion of \( v \) (and the edges incident with \( v \)) from \( G \). If \( E' \subseteq E(G) \) and \( V' \subseteq V(G) \), then \( G\setminus E' \), \( G/E' \), and \( G - V' \) are defined in the obvious way. If \( H \) is a subgraph of \( G \), then let \( G/H = G/E(H) \), and let \( G \setminus H = (G\setminus E(H)) - V_H \), where \( V_H \) is the set isolated vertices in \( G\setminus E(H) \) whose elements are not isolated vertices in \( G \).

If \( H \) can be obtained by deleting only vertices from \( G \), then it is standard to say that \( H \) is an induced subgraph of \( G \). If \( V' \subseteq V(G) \), then \( G[V'] \) is the induced subgraph obtained by deleting \( V(G) - V' \) from \( G \). If \( E' \subseteq E(G) \), then \( G[E'] \) is the smallest subgraph of \( G \) whose set of edges is \( E' \). We shall use the notation \( H \leq_G G \) to denote that \( H \) is a subgraph of \( G \). We say that \( H \) is a topological minor of \( G \), denoted \( H \leq_t G \), if some subdivision of \( H \) is a subgraph of \( G \). We write \( G \cong H \) to indicate that the graphs \( G \) and \( H \) are isomorphic.

We say that a graph \( G \) is 2-connected if \( G \) is loopless, \( |V(G)| + |E(G)| \geq 4 \), and \( G - v \) is connected, for each \( v \in V(G) \). Equivalently, a graph \( G \) is 2-connected if and only if \( |E(G)| \geq 2 \) and each pair of edges of \( G \) is contained in a cycle of \( G \). Also, we say that \( G \) is 3-connected if \( G \) is loopless, \( |V(G)| \geq 4 \), and \( G - \{u,v\} \) is connected, for each pair \( \{u,v\} \subseteq V(G) \). By a block of a graph \( G \), we mean an isolated vertex of \( G \), a loop of \( G \), a cut-edge of \( G \), or a maximal 2-connected subgraph of \( G \).

Let \( H \) be a subgraph of a graph \( G \). Then a bridge of \( H \) in \( G \) is one of the following kinds of subgraphs of \( G \):

(i) An edge of \( E(G) - E(H) \) contained in \( G[V(H)] \).

(ii) The union of a component \( C \) of \( G - V(H) \) and the set of edges that have one vertex in \( V(C) \) and the other vertex in \( V(H) \).

The disjoint union of sets or graphs will be denoted by \( \biguplus \). If \( k \) is a positive integer, then let \([k]\) denote the set of nonnegative integers less than \( k+1 \), and let \([k]_+ \) denote the set of positive integers less than \( k+1 \).

2. Preliminary results

We begin by describing graphs of very small type. The following self-evident theorem characterizes graphs of type zero, one, and two.

2.1. Theorem. Suppose \( G \) is a graph. The type of \( G \) is

(i) zero if and only if \( G \) is edgeless;
(ii) at most one if and only if \( G \) has no cycles other than loops; and
(iii) at most two if and only if every block of \( G \) is a multi-edge, a cycle, or an isolated vertex. \( \square \)

As we remarked in Section 1, type is not monotone under the taking of minors. As a matter of fact, it is not monotone even under the taking of induced subgraphs. Consider the graph \( D \) in Figure 1. The induced subgraph \( D - v \) is isomorphic to \( C_{5,3}^* \). We shall see later in Lemma 2.4 that \( t(C_{5,3}^*) = 5 \), but now we show that \( t(D) \leq 4 \). If the edges \( e \) and \( f \) are contracted in \( D \), then the resulting graph consists of five 3-cycles meeting at a single vertex. Since each block of \( D/\{e, f\} \) is a 3-cycle, \( t(D/\{e, f\}) = 2 \), by Theorem 2.1. Thus \( t(D) \leq t(D/\{e, f\}) + |\{e, f\}| = 2 + 2 = 4 \).

\[ \text{Figure 1. } D \text{ is the union of } C_{5,3}^*, e, \text{ and } f. \]

The graph in Figure 1 can be easily modified to show that this lack of monotonicity under the taking of induced subgraphs is arbitrarily “bad” in the sense that there are graphs \( G \) and \( H \) such that \( G \) is an induced subgraph of \( H \), but \( t(G) = t(H) \) is arbitrarily large.

Although type does not have monotonicity under the taking of induced subgraphs, it does have some very special kinds of monotonicity that we shall describe below. Let \( G \) be a graph, The simplification of \( G \), denoted \( \tilde{G} \), is obtained by deleting the loops of \( G \) and by replacing each proper multi-edge of \( G \) with a link-edge. Now, let \( \mathcal{C} \) be the collection of cycles in \( G \) each element of which has at most one vertex of degree exceeding two in \( G \), and let \( \mathcal{P} \) be the collection of proper paths \( P \) in \( G \) such that each internal vertex of \( P \) has degree 2 in \( G \). Then the cosimplification of \( G \) is obtained by contracting all but one edge of each element of \( \mathcal{C} \) and all but one edge of each maximal element of \( \mathcal{P} \) in \( G \).

A graph \( G \) is simpler than a graph \( H \) if \( G \) is a proper subgraph of \( H \), and the simplifications of \( G \) and \( H \) are isomorphic. A graph \( G \) is cosimpler than a graph \( H \) if \( G \) can be obtained by contracting a non-empty set of edges in \( H \), and the cosimplifications of \( G \) and \( H \) are isomorphic; equivalently, \( G \) is cosimpler than \( H \) if \( H \) can be obtained by subdividing each edge in a non-empty subset of \( E(G) \) with at least one new vertex.

2.2. Lemma. If \( G \) is simpler or cosimpler than \( H \), then \( t(G) \leq t(H) \).

Proof. We shall only consider the case when \( G \) is simpler than \( H \); the proof in the other case is very similar and left for the reader. Clearly, we may assume that \( |E(H)| = |E(G)| + 1 \). We proceed by induction on \( t(H) \).

If \( t(H) = 1 \), then the claim follows from Theorem 2.1. Suppose now that \( t(G') \leq t(H') \) whenever \( G' \) is a graph that is simpler than \( H' \) with one edge fewer than \( H' \)
and $t(H') < t(H)$. Let $e$ denote the edge of $H$ that is not in $G$. The proof in the case when $e$ is a loop is trivial. Hence we may assume that $e$ is parallel to some edge $e_G$ of $G$. Let $B_H$ denote the block of $H$ containing $e$ and $e_G$, and let $B_G$ denote the block of $G$ containing $e_G$. Since each block in $G$ different from $B_G$ is a block in $H$, it is sufficient to show that $t(B_G) \leq t(B_H)$. Let $e'$ be an edge of $B_H$ such that $t(B_H) = \min\{t(B_H \setminus e'), t(B_H / e')\} + 1$. Then $t(B_H \setminus e') < n$ or $t(B_H / e') < n$. If $e'$ is not parallel to $e$, then $B_G / e'$ and $B_G / e'$ are smaller than $B_H \setminus e'$ and $B_H / e'$, respectively. Since $t(B_H \setminus e') < n$ or $t(B_H / e') < n$, it follows from the induction hypothesis that $t(B_G \setminus e') \leq t(B_H \setminus e')$ or $t(B_G / e') \leq t(B_H / e')$, and so $t(B_G) \leq t(B_H)$. Hence we may assume that $e' = e$ or $e'$ is parallel to $e$. If $t(B_H) = t(B_H \setminus e') + 1$, then $B_H \setminus e'$ is isomorphic to $B_G$, and consequently, $t(B_G) < t(B_H)$. If $t(B_H) = t(B_H / e') + 1$, then $B_G / e'$ is isomorphic to $B_H / e'$, and we conclude from the induction hypothesis that $t(B_G / e') \leq t(B_H / e')$. Hence, $t(B_G) \leq t(B_H)$ and thus $t(G) \leq t(H)$.

Now we turn our attention to basic graphs of large type: fans, multicycles and comulticycles.

2.3. Lemma. For every positive integer $n$, the type of the $n$-fan is $[\log_2 n] + 1$.

Proof. Let us consider the augmented $n$-fan $F'_n$, which is the graph obtained by adding an edge $f'_n$ that is parallel to $f_n$, where $f_n$ is the edge of $F_n$ as illustrated in Figure 2.

![Figure 2](image)

Note that, by Lemma 2.2, we have $t(F_n) \leq t(F'_n) \leq t(F_{n+1})$, for each positive integer $n$, since $F_n$ is simpler than $F'_n$, and $F'_n$ is cosimpler than $F_{n+1}$. In particular, it follows that $t(F_n) \leq t(F'_m) \leq t(F'_n)$ whenever $m$ and $n$ are positive integers and $m \leq n$.

We shall proceed by induction on $n$. In addition to showing that $t(F_n) = [\log_2 n] + 1$, we shall also show that $t(F'_n) = [\log_2 (n + 1)] + 1$.

If $n = 1$, the proof is clear. Now, assume that $n$ is an integer exceeding 1 and that $t(F_n) = [\log_2 n'] + 1$ and $t(F'_n) = [\log_2 (n' + 1)] + 1$, for each positive integer $n'$ less than $n$.

First, we show that $t(F_n) \leq [\log_2 n] + 1$ and $t(F'_n) \leq [\log_2 (n + 1)] + 1$. Consider $F_n \setminus e[\frac{n}{2}]$ and $F'_n \setminus e[\frac{n}{2}]$. The graph $F_n \setminus e[\frac{n}{2}]$ consists of a block that is isomorphic to $F[\frac{n}{2}]$, and another block that is isomorphic to $F[\frac{n}{2}]$. It follows that $t(F_n) \leq t(F[\frac{n}{2}]) + 1 = (\log_2 [\frac{n}{2}]) + 1 = \log_2 n + 1$. Similarly, $F'_n \setminus e[\frac{n}{2}]$ consists of a block that is isomorphic to $F[\frac{n}{2}]$ and another block that is isomorphic to $F[\frac{n}{2}]$. If $n$ is even, then $F[\frac{n}{2}] = F[\frac{n}{2}]$ is simpler than $F'[\frac{n}{2}] = F'[\frac{n}{2}]$. It follows that $t(F_n) \leq$
\[(F_n^2) + 1 = (\lfloor \log_2 \left( \frac{n}{2} + 1 \right) \rfloor + 1) + 1 = (\lfloor \log_2 (n+2) + 1 \rfloor + 1) + 1 = \lfloor \log_2 (n+1) \rfloor + 1,\]
when \(n\) is even. On the other hand, if \(n\) is odd, then \(F_n^2 = F_{n+2}\) is cosimpler than \(F_{\frac{n+3}{2}}\). It follows that if \(n\) is odd, then \(t(F_n) = t(F_{n+2}) = (\lfloor \log_2 \left( \frac{n+1}{2} \right) \rfloor + 1) + 1 = (\lfloor \log_2 (n+1) - 1 \rfloor + 1) + 1 = \lfloor \log_2 (n+1) \rfloor + 1.

It remains to show that \(t(F_n) \geq \lfloor \log_2 n \rfloor + 1\) and \(t(F_n^2) \geq \lfloor \log_2 (n+1) \rfloor + 1\).

Since the proofs of these inequalities are very similar, we only prove the first one, while leaving the other to the reader. If \(e_i, f_i,\) or \(f_i\) is deleted from \(F_n\), then the resulting graph consists of two blocks, one of which is isomorphic to \(F_{n'}\), for some integer \(n'\) satisfying \(\frac{n}{2} \leq n' < n\), and so \(t(F_{n'}) \geq t(F_{\frac{n}{2}}) = \lfloor \log_2 \left( \frac{n}{2} \right) \rfloor + 1 = \lfloor \log_2 n \rfloor\). Thus \(t(F_n \setminus e_i) \geq \lfloor \log_2 n \rfloor\) and \(t(F_n \setminus f_i) \geq \lfloor \log_2 n \rfloor\) for each \(i \in [n-1]^+\) and for each \(j \in \{1,n\}\). If \(f_i\) is deleted from \(F_n\), where \(i\) is an integer satisfying \(1 < i < n\), then \(F_{n-1}\) is cosimpler than the resulting graph \(F_n \setminus f_i\); hence, \(t(F_n \setminus f_i) \geq t(F_{n-1}) = \lfloor \log_2 (n-1) \rfloor + 1 \geq \lfloor \log_2 n \rfloor\), for each \(i\) satisfying \(1 < i < n\).

If \(e_i, f_i,\) or \(f_i\) is contracted in \(F_n\), where \(i \in [n-1]^+\), then \(F_{n-1}\) is simpler than the resulting graph \(F_n / e_i\), where \(e \in \{e_i : i \in [n-1]^+\} \cup \{f_i, f_i\}\), and hence \(t(F_n / e_i) \geq t(F_{n-1}) \geq \lfloor \log_2 n \rfloor\), for each \(e \in \{e_i : i \in [n-1]^+\} \cup \{f_i, f_i\}\). If \(f_i\) is contracted in \(F_n\), where \(i\) is an integer satisfying \(1 < i < n\), then the resulting graph \(F_n / f_i\), consists of two blocks, one of which is isomorphic to \(F_{n'}\), for some integer \(n'\) satisfying \(\frac{n}{2} \leq n' < n\). Hence, \(t(F_n / f_i) \geq t(F_{\frac{n}{2}}) = \lfloor \log_2 \left( \frac{n}{2} \right) \rfloor + 1 \geq \lfloor \log_2 n \rfloor\), for each integer \(i\) satisfying \(1 < i < n\). Thus, \(t(F_n / e_i) \geq \lfloor \log_2 n \rfloor\) and \(t(F_n / e_i) \geq \lfloor \log_2 n \rfloor\), for each \(e \in E(F_n)\). Consequently, the induction hypothesis implies that \(t(F_n) \geq \lfloor \log_2 n \rfloor + 1\) for any positive integer \(n\).

\[\square\]

2.4. Lemma. For every integer \(n\) exceeding 3, each of the \((n,n-2)\)-multicycle and the \((n, n-2)\)-comulticycle has type \(n\).

Proof. We prove a more general statement regarding the type of \(C_{n,m}\), where \(m\) and \(n\) are integers exceeding 1 and 3, respectively, and \(C_{n,m}\) represents any graph obtained by replacing one edge of \(C_n\) with a multi-edge containing exactly \(m\) edges and by replacing each of the remaining edges of \(C_n\) with a multi-edge containing at least \(m\) edges. The result that we prove here is that \(t(C_{n,m}) = \min\{m, n+2\}\).

It will follow immediately that \(t(C_{n,n-2}) = n\), for each integer \(n\) exceeding 3.

Let \(E_1\) be a multi-edge of \(C_{n,m}\) consisting of \(m\) edges. Then each block of \(C_{n,m} / E_1\) is a multi-edge. It follows from Theorem 2.1 that \(t(C_{n,m} / E_1) \leq 2\); hence, \(t(C_{n,m}) \leq |E_1| + t(C_{n,m} / E_1) \leq m + 2\). Let \(E_2\) be a set of \(n-2\) edges of \(C_{n,m}\) such that if \(f\) and \(f'\) are distinct edges in \(E_2\), then \(f\) and \(f'\) belong to distinct multi-edges of \(C_{n,m}\). Then \(C_{n,m} / E_2\) consists of a block that is a multi-edge and blocks that are loops. It follows from Theorem 2.1 that \(t(C_{n,m} / E_2) \leq 2\); hence, \(t(C_{n,m}) \leq |E_2| + t(C_{n,m} / E_2) \leq n\). Thus, \(t(C_{n,m}) \leq \min\{n, m+2\}\).

To prove the opposite inequality, we proceed by induction on \(m\) and \(n\). It is easy to check that \(t(C_{n,2}) = t(C_{n,m}) = 4\) for integers \(m\) and \(n\) exceeding, respectively, 1 and 3. Now, let us assume that if \(m'\) and \(n'\) are integers satisfying \(2 \leq m' < m\) and \(4 \leq n' < n\), then \(t(C_{n,m'}) = \min\{m' + 2\}\), and \(t(C_{n',m}) = \min\{n', m + 2\}\). Consider the graph \(C_{n,m}\). For any edge \(e \in E(C_{n,m})\), the graph \(C_{n,m} / e\) is the union of a block that is a graph \(C_{n-1,m}\) and blocks that are loops. It follows from the induction hypothesis that \(t(C_{n-1,m}) = \min\{n-1, m + 2\}\). If \(e\) belongs to a multi-edge of \(C_{n,m}\) that contains exactly \(m\) edges, then \(C_{n,m} / e\) is a graph \(C_{n,m-1}\). It follows from the induction hypothesis that \(t(C_{n,m-1}) = \min\{n, m + 1\}\). If \(e\) belongs to a multi-edge of \(C_{n,m}\) that contains more than \(m\)
edges and \( f \) belongs to a multi-edge of \( C_{n,m} \) that contains exactly \( m \) edges, then \( C_{n,m} \setminus \{ e, f \} \) is a graph \( C_{n,m-1} \) that is simpler than \( C_{n,m} \setminus e \). By Theorem 2.1 and the induction hypothesis, \( t(C_{n,m} \setminus e) \geq t(C_{n,m-1}) = \min\{n, m + 1\} \) when \( e \) belongs to a multi-edge containing more than \( m \) edges. It follows that \( t(C_{n,m}) = \min\{\min\{n-1, m+2\}, \min\{n, m+1\}\} + 1 = \min\{n-1, m+1\} + 1 = \min\{n, m+2\} \), as required.

The proof for conmulticycles can be easily obtained by using the concept of duality. We leave the details to the reader. 

\( \square \)

3. 3-CONNECTED GRAPHS

Although type is not monotone under the taking of minors, we can describe 3-connected graphs of large type as those without large fan minors. This characterization appears as Theorems 1.4 and 1.5. The entire remainder of this section will be devoted to proving these results. We start with some lemmas.

**3.1. Lemma.** If \( n \) is an integer exceeding 1, and \( F_n \leq_m G \), then \( G \) contains a vertex set \( S = \{v_i : i \in [n]_+\} \), a \( v_1v_n \)-path \( P \), and a tree \( T \) whose set of leaves is \( S \), such that \( P \cap T = S \) and \( F_n \leq_m P \cup T \).

**Proof.** If \( F_n \leq_m G \), then there are disjoint subsets \( E_d \) and \( E_c \) in \( E(G) \) such that \( F_n \cong (G \setminus E_d/E_c)_E \), where \( H_E \) denotes the subgraph of a graph \( H \) obtained by deleting all isolated vertices from \( H \); equivalently, \( H_E = H[E(H)] \). Among all pairs \( (E_d, E_c) \) of disjoint sets of edges of \( E(G) \) such that \( F_n \cong (G \setminus E_d/E_c)_E \), choose one for which \( |E_c| \) is minimum, and denote this pair by \( (D, C) \). Let \( G' = (G \setminus D)_E \). A typical \( G' \) is illustrated in Figure 3, where the dashed edges and the solid edges form, respectively, a path \( P \) and a tree \( T \), and \( F_8 \) is obtained by contracting the unshaded edges.

![Figure 3. A typical \( G' \) that contains an \( F_8 \)-minor.](image-url)

If \( C \) is empty, then \( G' \cong F_n \), and \( S, P, \) and \( T \) are obvious. So, we may assume that \( C = \{e_i : i \in [k]_+\} \), for some positive integer \( k \). Let the sequence \( (e_i)_{i=1}^k \) be an arbitrary ordering of the elements of \( C \). Let \( G_0 = G' \), and, inductively, let \( G_i = (G_{i-1}/e_i)_E \) for each \( i \in [k]_+ \), so that \( G_k \cong F_n \). Note that since \( G_k \) is a block, \( E(G_k) \) is contained in a single block \( B_i \) of \( G_i \), for each \( i \in [k]_+ \). Moreover, \( G_i \) is a block for each \( i \in [k]_+ \), that is, \( G_i = B_i \). This can be seen as follows. If \( G_i \) contained a block \( B'_i \neq B_i \), then \( (D \cup E(B'_i), C - E(B'_i)) \) would be a pair of disjoint sets of edges such that \( (G \setminus (D \cup E(B'_i)))/(C - E(B'_i)))_E \cong F_n \), but, since \( G_i \) has no isolated
vertices (hence, \(E(B'_i)\) is non-empty), \(|C - E(B'_i)| < |C|\): a contradiction to the minimality of \(C\).

Now, we show that \(G' = P \cup T\). We proceed by induction on \(j\) to prove that, for each \(j \in [k]\), the graph \(G_{k-1}\) contains a \(v_1 v_n\)-path \(P_{k-1}\) and a tree \(T_{k-1}\) whose set of leaves is \(S\), such that \(P_{k-1} \cap T_{k-1} = S\). If \(j = 0\), then \(k - j = k\), and, since \(G_k \cong F_n\), it is obvious what \(S, P_k, \) and \(T_k\) are. Assume that \(G_{k-1}\) contains subgraphs \(P_{k-1}\) and \(T_{k-1}\) that have the required properties, for each nonnegative integer \(j < k\), and let \(i = k - j\). Let \(v\) denote the vertex of \(G_i\) obtained by contracting \(e_i\) in \(G_{i-1}\).

First, if \(v \in V(P_i) - S\), then \(v\) is incident in \(G_i\) with exactly two edges \(e\) and \(f\), which lie in \(P_i\). After expanding \(v\) in \(G_i\) to \(e_i\) to obtain \(G_{i-1}\), since \(G_{i-1}\) is a block, \(e_i\) is neither a loop nor a cut-edge in \(G_{i-1}\). It follows that \(G_{i-1}[E(P_i) \cup e_i]\) is a \(v_1 v_n\)-path that contains the subpath \(ee_i f\). Let \(P_{i-1} = G_{i-1}[E(P_i) \cup e_i]\) and \(T_{i-1} = T_i\). It is straightforward that \(P_{i-1}\) and \(T_{i-1}\) have the required properties.

Now, if \(v \in V(T_i) - S\), then \(v\) is incident in \(G_i\) with only edges in \(T_i\). After expanding \(v\) in \(G_i\) to \(e_i\) to obtain \(G_{i-1}\), since \(G_{i-1}\) is a block, \(e_i\) is neither a loop nor a cut-edge. Hence \(G_{i-1}[E(T_i) \cup e_i]\) is a tree whose set of leaves is \(S\). Let \(P_{i-1} = P_i\) and \(T_{i-1} = G_{i-1}[E(T_i) \cup e_i]\). It follows that \(P_{i-1}\) and \(T_{i-1}\) have the required properties.

Finally, assume that \(v \in S\). Then either \(v\) is not an endvertex of \(P\), or \(v \in \{v_1, v_n\}\). We consider the case in which \(v\) is not an endvertex of \(P\); the proof when \(v \in \{v_1, v_n\}\), which is very similar, is left for the reader. If \(v\) is not an endvertex of \(P\), then \(v\) is incident in \(G_i\) with exactly three edges \(e, f\), and \(g\), where \(\{e, f\} \subseteq E(P_i)\) and \(g \in E(T_i)\). After expanding \(v\) in \(G_i\) to \(e_i\) to obtain \(G_{i-1}\), since \(G_{i-1}\) is a block, \(e_i\) is neither a loop nor a cut-edge. It follows that one vertex of \(e_i\) is trivalent in \(G_{i-1}\); call this vertex \(v\). Then one of the following holds for \(G_{i-1}\):

(i) \(G_{i-1}[E(P_i) \cup e_i]\) is a \(v_1 v_n\)-path that contains the subpath \(ee_i f\), in which case, \(P_{i-1} = G_{i-1}[E(P_i) \cup e_i]\) and \(T_{i-1} = G_{i-1}[E(T_i)]\) have the required properties.

(ii) \(G_{i-1}[E(T_i) \cup e_i]\) is a tree whose set of leaves is \(S\), in which case, \(P_{i-1} = P_i\) and \(T_{i-1} = G_{i-1}[E(T_i) \cup e_i]\) have the required properties.

If \(v \in \{v_1, v_n\}\), then \(v\) is incident in \(G_i\) with exactly two edges \(e \in E(P_i)\) and \(f \in E(T_i)\). If \(v\) is expanded in \(G_i\) to \(e_i\) to obtain \(G_{i-1}\), then \(e_i f\) is a subpath in \(G_{i-1}\). Let \(v\) denote the vertex in \(G_{i-1}\) common to \(e_i\) and \(f\). It follows that the graphs \(P_{i-1} = G_{i-1}[E(P_i) \cup e_i]\) and \(T_{i-1} = T_i\) have the required properties.

3.2. Lemma. If \(n\) is an integer exceeding 1, and \(F_n \preceq_m G\), then \(F_{\lfloor \frac{n}{2} \rfloor} \preceq_m G \setminus e\) for every \(e \in E(G)\).

Proof. Assume that \(F_n \preceq_m G\). Then there are subgraphs \(P\) and \(T\) of \(G\) that satisfy the requirements specified in Lemma 3.1. If \(e \notin P \cup T\), then \(P \cup T\) is a subgraph of \(G \setminus e\), and hence \(F_{\lfloor \frac{n}{2} \rfloor} \preceq_m F_n \preceq_m P \cup T \preceq_m G \setminus e\). If \(e \in P\), then one component \(P'\) of \(P \setminus e\) is a subpath of \(P\) containing at least \(\left\lceil \frac{n}{2} \right\rceil\) vertices in \(S\), and hence \(F_{\lfloor \frac{n}{2} \rfloor} \preceq_m P' \setminus T \preceq_m G \setminus e\). If \(e \in T\), then one component \(T'\) of \(T \setminus e\) is a subgraph of \(T\) containing at least \(\left\lceil \frac{n}{2} \right\rceil\) vertices in \(S\), and hence, \(F_{\lfloor \frac{n}{2} \rfloor} \preceq_m P \setminus T' \preceq_m G \setminus e\).

3.3. Lemma. If \(n\) is an integer exceeding 1, and \(F_n \preceq_m G\), then \(F_{\lfloor \frac{n}{2} \rfloor} \preceq_m G / e\) for every \(e \in E(G)\).

Proof. Assume that \(F_n \preceq_m G\). Then there are subgraphs \(P\) and \(T\) of \(G\) that satisfy the requirements stated in Lemma 3.1. If \(e\) is a loop, or if some vertex of \(e\) does not
lie in $P \cup T$, then $F_{\lfloor \frac{n}{2} \rfloor} \leq_s F_n \leq_m P \cup T \leq_s G/e$. So we may assume that $e \parallel xy$, and $x$ and $y$ are distinct vertices of $P \cup T$.

We shall use the following notation in proving this lemma. If $\{u, v\} \subseteq V(P)$, then let $P_{u,v}$ denote the $uv$-subpath of $P$, and, for each $v \in S$, let $e_v$ denote the edge of $T$ incident with $v$. We shall consider several cases depending on the location of $x$ and $y$.

Suppose first that $\{x, y\} \subseteq S$. Let $P'$ be the element of $\{P_{xy} \cup e, (P \setminus P_{xy}) \cup e\}$
that contains at least as many vertices of $S$ as the other, and let $S' = S \cap V(P')$. Let $m = |S'|$. Clearly, $m \geq \left\lceil \frac{n}{2} \right\rceil + 1$. Contract $e$ to $x$. It follows that $P'/e$ contains a path $P''$ having $m - 1$ vertices of $S' - y$. Since $(G/e)[E(T)]$ is obtained from $T$ by identifying $x$ and $y$, it follows that $\bigcup_{e \in S' - y} e_v$ is acyclic in $G/e$, and, hence, there is a tree $T'$ in $(G/e)[E(T)]$ that contains $\bigcup_{e \in S' - y} e_v$. Furthermore, $P'' \cap T' = S' - y$, and the set of leaves of $T'$ contains $S' - y$. It follows that $F_{m-1} \leq_m G/e$, and, since $m - 1 \geq \left\lceil \frac{n}{2} \right\rceil$, that $F_{\lfloor \frac{n}{2} \rfloor} \leq_m F_{m-1} \leq_m P'' \cup T' \leq_s G/e$.

The case when $\{x, y\} \subseteq P$, but $\{x, y\} \cap S \leq 1$ is very similar to the one presented above, so we leave the details to the reader.

Suppose now that that $e \parallel xy$ has both vertices in $T - S$. Then $P$ is a path in $G/e$. It is clear that $\bigcup_{e \in S} e_v$ is acyclic in $G/e$, and, since $(G/e)[E(T) - e]$ is connected, $(G/e)[E(T) - e]$ contains a tree $T'$ whose set of leaves is $S = P \cap T'$. It follows that $F_{\lfloor \frac{n}{2} \rfloor} \leq_s F_n \leq_m P \cup T' \leq_s G/e$.

It remains to consider the case when one of $x$ and $y$, say $x$, is in $P$ and the other, $y$, is in $T - S$. Then there are not more than four edges of $P \cup T \cup e$ incident with $x$. One of these edges is $e$, and there are two distinct edges $e'^* \in f^*$ of $P' \cup T \cup e$ incident with $x$. Consider the graph $(P \cup T \cup e / e \{e'^*, f^*\})$. One component $P'$ of $(P \cup T \cup e / e \{e'^*, f^*\})$ is a path that contains at least $\left\lceil \frac{n}{2} \right\rceil$ vertices of $P$. Let $S'$ denote $V(P') \cap S$. It follows that $\bigcup_{e \in S'} e_v$ is acyclic in $G/e$. Since $(P \cup T \cup e / e \{e'^*, f^*\})$ is connected, $(P \cup T \cup e / e \{e'^*, f^*\})$ contains a tree $T'$ whose set of leaves is $S' = P' \cap T'$. Then $F_{\lfloor \frac{n}{2} \rfloor} \leq_m P' \cup T' \leq_s (P \cup T \cup e / e \{e'^*, f^*\}) \leq_s G/e$.

Now we are ready to prove Theorem 1.4

Proof of Theorem 1.4. Suppose the theorem fails and let $\mathcal{S}$ be the collection of counterexamples to Theorem 1.4, that is, $\mathcal{S}$ consists of graphs $H$ for which there is a positive integer $n(H)$ such that $F_{n(H)} \leq_m H$, but $t(H) < \lfloor \log_2 n(H) \rfloor + 1$. Let $\mathcal{S}_0$ be the subcollection of $\mathcal{S}$ each of whose elements contain a minimum number of edges and no isolated vertices, and let $n = \min\{n(H) : H \in \mathcal{S}_0\}$. Then $\mathcal{S}_0$ contains a graph $G$ such that $n(G) = n$. Any such $G$ is a minimal counterexample to Theorem 1.4 in the sense defined above. Note that $n \geq 2$ since if $F_1$ is a minor of a graph $H$, then $t(H) \geq \lfloor \log_2 1 \rfloor + 1 = 1$. It follows that since $F_n \leq_m G$, there are subgraphs $P$ and $T$ of $G$ that have the properties specified in Lemma 3.1.

The minimality of $G$ implies that $G$ is a block, and hence $\min\{t(G\setminus e), t(G/e)\} = t(G) - 1$. First, suppose that there is an edge $e$ such that $t(G\setminus e) = t(G) - 1$. By Lemma 3.2, $F_{\lfloor \frac{n}{2} \rfloor} \leq_m G/e$. Since $G\setminus e$ is not a counterexample,

$$t(G\setminus e) \geq \lfloor \log_2 \left\lfloor \frac{n}{2} \right\rfloor \rfloor + 1 \geq \lfloor \log_2 \frac{n}{2} \rfloor + 1 = \lfloor \log_2 n - 1 \rfloor + 1 = \lfloor \log_2 n \rfloor,$$

and hence $t(G) \geq \lfloor \log_2 n \rfloor + 1$; a contradiction. Thus, if $G$ is a counterexample, then there is an edge $e$ such that $t(G/e) = t(G) - 1$.

Let $e$ be an edge such that $t(G/e) = t(G) - 1$. By Lemma 3.3, $F_{\lfloor \frac{n}{2} \rfloor} \leq_m G/e$, that is, $F_{\lfloor \frac{n}{2} \rfloor} \leq_m G/e$ if $n$ is even, and $F_{n-1} \leq_m G/e$ if $n$ is odd. Since $G/e$ is not a
counterexample,
\[ t(G/c) \geq \begin{cases} 
\left\lfloor \log_2 \frac{n}{2} \right\rfloor + 1 = \left\lfloor \log_2 n \right\rfloor, & \text{if } n \text{ is even; and} \\
\left\lfloor \log_2 \frac{n-1}{2} \right\rfloor + 1 = \left\lfloor \log_2 (n-1) \right\rfloor = \left\lfloor \log_2 n \right\rfloor, & \text{if } n \text{ is odd.}
\end{cases} \]
Hence, \( t(G) \geq \left\lfloor \log_2 n \right\rfloor + 1 \); a contradiction, which proves Theorem 1.4.

We now focus on proving Theorem 1.5. One of the major tools in proving this theorem is the following result of Seymour [6] (see also [1]).

3.4. **Theorem.** Let \( C \) be a largest circuit of a connected matroid \( M \). Then the size of every circuit of \( M/C \) is less than \( C \).

We use this theorem to derive the following two corollaries, the first of which is evident.

3.5. **Corollary.** Every two longest cycles in a 2-connected graph intersect in at least two vertices.

3.6. **Corollary.** If the type of a 2-connected graph \( G \) exceeds \( \frac{N(N+1)}{2} \), for some integer \( N \) greater than 1, then \( G \) contains a cycle of length more than \( N \).

**Proof.** Let \( G \) be a 2-connected graph, and assume that a longest cycle \( C \) of \( G \) has length \( N \). We shall show that the type \( t(G) \) of \( G \) is at most \( \frac{N(N+1)}{2} \) by induction on \( N \).

If \( N = 2 \), then \( G \) is multi-edge and, by Theorem 2.1, the claim is immediate. Now, assume that \( N > 2 \) and that if the length of a longest cycle in a 2-connected graph \( G' \) is \( N' \), for some \( N' < N \), then \( t(G') \leq \frac{N(N'+1)}{2} \). After contracting \( C \) in \( G \), every cycle of \( G/C \) has length less than \( N \), by Theorem 3.4. In particular, each cycle of each block of \( G/C \) has length less than \( N \). By hypothesis, the type of each block that is neither a loop nor a cut-edge of \( G/C \) does not exceed \( \frac{(N-1)N}{2} \), and it is evident that the type of a block that is a loop or a cut-edge does not exceed \( \frac{(N-1)N}{2} \). It follows that \( t(G) \leq N + \frac{(N-1)N}{2} = \frac{N(N+1)}{2} \), as required, since \( G/C \) was obtained by contracting \( N \) elements in \( G \), and since \( t(G/C) \leq \frac{(N-1)N}{2} \).

The second important ingredient of the proof of Theorem 1.5 is the following result.

3.7. **Theorem.** For each positive integer \( n \) exceeding 2 there is an integer \( N \) such that if \( G \) is a 3-connected graph with a circuit on \( N \) vertices, then the \( n \)-wheel, \( W_n \), is a minor of \( G \).

Although, to our knowledge, this theorem has not been explicitly stated in literature, there are a few papers from which its proof can be derived. Since showing such a derivation formally here would require a large amount of new terminology and notation, we instead refer the reader to two proofs in [3] and [2]. The first of these, the proof of (1.4) in [3], speaks of graphs, although the derivation of Theorem 3.7 from it is fairly technical. On the other hand, the derivation of Theorem 3.7 from the proof of Theorem (1.5) in [2] requires translating from the language of binary matroids to the language of graphs, but the technical details of the derivation are easier.

It is worth noting that the value of \( N \) as a function of \( n \) that can be obtained through either derivation is extremely large and believed to be very far from the best possible bound. We are now ready to present the proof of Theorem 1.5.
Proof of Theorem 1.5. For each integer exceeding 2, let \( t_n = \frac{N(N-1)}{2} + 1 \), where \( N \) is the number depending on \( n \) from Theorem 3.7. If \( G \) is a 3-connected graph whose type is at least \( t_n \), then, by Theorem 3.4, \( G \) contains a cycle of length at least \( N \). The conclusion now follows immediately from Theorem 3.7.

4. 2-SUMS AND TREE STRUCTURES

The remainder of the paper will be devoted to proving Theorem 1.7. The main idea of the proof is to decompose the graph into pieces that are either 3-connected, or have very simple structure. We shall use a decomposition that relies on a result of Tutte, which states that every 2-connected graph has a canonical decomposition into simple 3-connected graphs, cycles, and multi-edges. In this section, we shall describe this decomposition and prove its basic properties, while in the remainder of the paper, we shall use it to prove Theorem 1.7.

If \( G \) is a graph, \( E_0 \) is a subset of \( E(G) \), and \( S \) is a set, then define a function \( L_G: E_0 \to S \times (V(G) \times V(G)) \): \( e \mapsto (s(e), (u(e), v(e))) \) so that for each \( e \) in \( E_0 \), \( u(e) \) and \( v(e) \) are the endvertices of \( e \), and if \( s(e) = s(f) \), then \( e = f \). Intuitively, we may think of \( L_G \) as a function which assigns to each edge \( e \) in \( E_0 \) a label \( s(e) \) and a direction where \( u(e) \) and \( v(e) \) are, respectively, the tail and the head of \( e \). Frequently, we shall describe these functions in this intuitive way. Also, it is convenient to think of the function \( L_G \) on \( E_0 \) as a partial function \( L_G: E(G) \to S \times (V(G) \times V(G)) \), where \( L_G(e) \) is defined if and only if \( e \in E_0 \); often, we shall consider such functions \( L_G \) without specifying the domain of definition. Call \( L_G \), a directed labeling of \( G \). It is clear that restricting the domain of \( L_G \) to a subset \( E' \subseteq E_0 \) results in a directed labeling \( L_G' \) of \( G \), which will be called a restriction of \( L_G \).

If the domain of \( L_G \) is the empty set, then call the directed labeling \( L_G \) of \( G \) trivial, and we may also say that \( G \) is unlabeled. It is also clear that if \( G' \) is a minor of \( G \), then \( L_G': E(G') \cap E_0 \to S \times (V(G') \times V(G')) \): \( e \mapsto (s(e), (u'(e), v'(e))) \) is a directed labeling of \( G' \), where \( u'(e) \) and \( v'(e) \) are the vertices in \( G' \) that correspond to \( u(e) \) and \( v(e) \), respectively, in \( V(G) \). In such case call \( L_G' \) the directed labeling of \( G' \) induced by \( L_G \).

Assume that \( L_H: E(H) \to S \times (V(H) \times V(H)) \): \( e \mapsto (s(e), (u_H(e), v_H(e))) \) and \( L_K: E(K) \to S \times (V(K) \times V(K)) \): \( e \mapsto (s(e), (u_K(e), v_K(e))) \) are directed labelings of disjoint graphs \( H \) and \( K \), respectively, and there is only one pair, \( h \in E(H) \) and \( k \in E(K) \), of edges such that \( s(h) = s(k) \). Then the edge-sum of \( H \) and \( K \) (with respect to \( L_H \) and \( L_K \), denoted \( (H, L_H) \bowtie_2 (K, L_K) \) or, more commonly, \( H \bowtie_2 K \)), is the graph defined as follows. If neither \( h \) nor \( k \) is a loop, then \( H \bowtie_2 K \) is obtained by first identifying \( h \) and \( k \) head-to-head and tail-to-tail, and then deleting the identified edge. If at least one of \( h \) and \( k \) is a loop, then \( H \bowtie_2 K \) is obtained by first contracting \( h \) to a vertex \( v_h \) and \( k \) to a vertex \( v_k \), and then identifying \( v_h \) and \( v_k \). We may sometimes refer to \( H \bowtie_2 K \) as the edge-sum of \( H \) and \( K \) along \( h \) and \( k \) when \( L_H \) and \( L_K \) are understood.

It is clear from the definition that edge-summing is commutative. Evidently, if \( H \) and \( K \) can be edge-summed along \( h \) and \( k \) (with respect to \( L_H \) and \( L_K \), then the edge set of \( H \bowtie_2 K \) is \( (E(H) - h) \cup (E(K) - k) \). It is easy to see that there is a partial function \( L_{H \bowtie_2 K}: E(H \bowtie_2 K) \to S \times (V(H \bowtie_2 K) \times V(H \bowtie_2 K)) \): \( e \mapsto (s(e), (u_{H \bowtie_2 K}(e), v_{H \bowtie_2 K}(e))) \), where \( u_{H \bowtie_2 K}(e) \) and \( v_{H \bowtie_2 K}(e) \) are the vertices in \( H \bowtie_2 K \) that correspond to the tail and head, respectively, of \( e \) determined by \( L_H \) or \( L_K \) (depending on whether \( e \) is in \( E(H) - h \) or in \( E(K) - k \)). Moreover, \( L_{H \bowtie_2 K} \)
is a directed labeling of $H \oplus K$ since $s(e) \neq s(f)$ for any two distinct edges $e$ and $f$ in $(E(H) - h) \cup (E(K) - k)$; we shall call $L_{H \oplus K}$ the directed labeling inherited from $L_H$ and $L_K$. If $L'_H$ and $L'_K$ are the directed labelings of, respectively, $H$ and $K$ obtained by reversing the directions assigned by $L_H$ and $L_K$ to the edges $h$ and $k$, then it is evident that $(H, L_H) \oplus_2 (K, L_K) = (H, L'_H) \oplus_2 (K, L'_K)$. We call this process of obtaining $L'_H$ and $L'_K$ from $L_H$ and $L_K$ pair direction reversal. If $h$ is not a block of $H$, and $k$ is not a block of $K$, then $H \oplus K$ is called the 2-sum of $H$ and $K$.

The following lemma is a well-known property of 2-sums.

4.1. Lemma. If $H$ and $K$ are 2-connected graphs that can be 2-summed along $h \in E(H)$ and $k \in E(K)$, then $H \oplus K$ is 2-connected. \hfill \Box

We have noted that edge-summing is commutative. In general, edge-summing is not associative, but there is “conditional” associativity. The condition that we must impose is that if $H$, $J$, and $K$ are pairwise disjoint graphs with directed labelings $L_H$, $L_J$, and $L_K$, respectively, then exactly two elements of $\{H \oplus J, H \oplus K, J \oplus K\}$ are defined.

Given a collection of pairwise disjoint graphs $\mathcal{G}$ on which we want to perform edge-sums, it is convenient to use a tree $T$ whose vertex set corresponds to $\mathcal{G}$ and whose edge set corresponds to a subset of the set of labels used in the directed labelings of the elements of $\mathcal{G}$. To avoid confusion between vertices and edges of elements of $\mathcal{G}$ and those of $T$, we shall call elements of $V(T)$ nodes and elements of $E(T)$ links. Moreover, Greek letters will be used to denote nodes and links of $T$, and Roman letters will be used to denote vertices and edges of elements of $\mathcal{G}$. We describe this correspondence between $\mathcal{G}$ and $T$ more precisely as follows.

Let $\mathcal{G} = \{G_i : i \in [n]\}$ be a collection of pairwise disjoint graphs, let $L_\mathcal{G} = \{L_{G_i} : i \in [n]\}$ be a collection of directed labelings of the elements of $\mathcal{G}$, and let $T$ be a tree on the node set $\{\xi_i : i \in [n]\}$, where $n$ is a nonnegative integer. Then $\mathcal{T} = (\mathcal{G}, L_\mathcal{G}, T)$ is an edge-sum tree if the following hold.

(i) If $\varepsilon = \xi_i \xi_j \in E(T)$, then there are precisely two graphs of $\mathcal{G}$, namely $G_i$ and $G_j$, each containing an edge labeled $\varepsilon$.

(ii) If $G_i \in \mathcal{G}$ has an edge labeled $\varepsilon$, then there is exactly one other graph $G_j \in \mathcal{G}$ in that has an edge labeled $\varepsilon$; moreover, $\xi_i \xi_j \in E(T)$.

It will be useful to look at a more general kind of tree structure (that includes the edge-sum trees) obtained by relaxing condition (ii). Call $\mathcal{T} = (\mathcal{G}, L_\mathcal{G}, T)$ a labeled edge-sum tree if and only if $\mathcal{G}$, $L_\mathcal{G}$, and $T$ are as above, and $\mathcal{T}$ satisfies condition (i) above and condition (ii)$'$ below.

(ii)$'$ If $G_i \in \mathcal{G}$ has an edge labeled $\varepsilon$, then there is at most one other graph $G_j \in \mathcal{G}$ that has an edge labeled $\varepsilon$, and if there is such a $G_j$, then $\xi_i \xi_j \in E(T)$.

If $\mathcal{T} = (\mathcal{G}, L_\mathcal{G}, T)$ is a labeled edge-sum tree, then call the elements of $\mathcal{G}$ the node graphs of $\mathcal{T}$, call $L_\mathcal{G}$ the directed labeling of $\mathcal{T}$, and call $T$ the tree of $\mathcal{T}$.

Given an edge-sum tree $\mathcal{T} = (\mathcal{G}, L_\mathcal{G}, T)$ and a subtree $T'$ of $T$, we can form the edge-sum tree $\mathcal{T}' = (\mathcal{G}', L_{\mathcal{G}'}, T')$, where $\mathcal{G}'$ is the subcollection of $\mathcal{G}$ corresponding to $V(T')$, by restricting the directed labeling associated with each element of $\mathcal{G}'$ in the appropriate way (that is, for each $G_i \in \mathcal{G}'$, there is an edge of $G_i$ labeled $\varepsilon$ if and only if $\varepsilon \in E(T')$ and $\xi_i$ is a vertex of $\varepsilon$). We shall say that $\mathcal{T}'$ is a restriction of $\mathcal{T}$ and that $\mathcal{T}'$ is the restriction of $\mathcal{T}$ induced by the subtree $T'$ of $T$. In particular, if
the subtree $T'$ is obtained by deleting a leaf $\xi$ from $T$, then we shall say that $T'$ is obtained by \textit{deleting} $\xi$ from $\mathcal{T}$ and let $\mathcal{T} - \xi$ denote $T'$.

A basic operation that we shall perform on a labeled edge-sum tree is forming its composition, which we define as follows. Given a labeled edge-sum tree $\mathcal{T} = (\mathcal{G}, L_T, T)$, we can obtain a graph $G(\mathcal{T})$ (with a directed labeling, that is, perhaps, trivial) called the \textit{composition} of $\mathcal{T}$, by edge-summing as dictated by the links of $T$ in the following manner. If $T$ has no links, then $T$ consists of a single node, $\mathcal{G}$ contains exactly one element, namely $G_0$, and there is nothing to do; hence $G(\mathcal{T}) = G_0$, and the edges of $G(\mathcal{T})$ are assigned labels and directions according to $L_{G_0}$. Inductively, if $E(T)$ is non-empty and $\xi = \xi_i \xi_j$ is a link of $T$, then form $\mathcal{T}' = (\mathcal{G}', L_{\mathcal{T}'}, T')$, where $\mathcal{G}'$ is obtained from $\mathcal{G}$ by replacing $G_i$ and $G_j$ with their edge-sum, $L_{\mathcal{T}'}$ is obtained from $L_\mathcal{G}$ by replacing $L_{G_i}$ and $L_{G_j}$ with the directed labeling $L_{G_i \oplus G_j}$ inherited from $L_{G_i}$ and $L_{G_j}$, and $T'$ is obtained from $T$ by contracting $\xi$ to a node $\xi$ that corresponds to $G_i \oplus G_j$. We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by \textit{contracting} $\xi$ in $\mathcal{T}$, and let $\mathcal{T} / \xi$ denote $\mathcal{T}'$. It is clear that $\mathcal{T}'$ is a labeled edge-sum tree. In particular, if $\mathcal{T}$ is an edge-sum tree, then so is $\mathcal{T}'$, and it follows that $G(\mathcal{T})$ is unlabeled. In general, when the directed labeling $L_\mathcal{T}$ of a labeled edge-sum tree $\mathcal{T} = (\mathcal{G}, L_\mathcal{T}, T)$ is understood, we shall let $(\mathcal{G}, T)$ denote $\mathcal{T}$. Also, we shall not indicate when edges of node graphs and compositions are assigned labels and directions except as needed.

It follows from the definition of the composition of a labeled edge-sum tree $\mathcal{T} = (\mathcal{G}, T)$ that there is a sequence $(\mathcal{T}_i)_{i=0}^n$ of labeled edge-sum trees where $T$ has $n$ links, $\mathcal{T}_0 = \mathcal{T}$, and $\mathcal{T}_i$ is obtained by contracting a link in $\mathcal{T}_{i-1}$, for each $i \in [n]_+$; it follows that $\mathcal{T}_n = (G(\mathcal{T}), K_1)$. Call each $\mathcal{T}_i$ in the above sequence a \textit{partial composition} of $\mathcal{T}$, and if $i \in [n-1]_+$, then the partial composition $\mathcal{T}_i$ is \textit{proper}. Such a sequence of partial compositions determines a natural way to edge-sum the elements of $\mathcal{G}$.

Figure 4 shows an edge-sum tree $\mathcal{T}$ and its composition $G(\mathcal{T})$. The nodes of the tree $T$ of $\mathcal{T}$ are indicated by the ovals, and the line segments that connect the ovals are the links of $T$. Each node graph of $\mathcal{T}$ is drawn inside its corresponding oval. The directed labeling of $\mathcal{T}$ assigns labels and directions to edges of the node graphs, as indicated. It follows that the line segment that connects the two nodes of $T$ whose node graphs each contain an edge labeled $\varepsilon_i$ is the link $\varepsilon_i$. The edges of $G(\mathcal{T})$ are the solid edges. For each $i \in [6]_+$, the dotted line segment labeled $i$ shows where two node graphs were edge-summed along the two edges labeled $\varepsilon_i$ (but it is not an edge of $G(\mathcal{T})$).

The terminology has been referring to \textit{the} composition (rather than a composition) $G(\mathcal{T})$ of a labeled edge-sum tree $\mathcal{T}$. Indeed, it is routine to verify that any composition of $\mathcal{T}$ results in a unique graph $G(\mathcal{T})$.

Let $\mathcal{T} = (\mathcal{G}, L_T, T)$ and $\mathcal{T}' = (\mathcal{G}', L_{\mathcal{T}'}, T')$ be directed edge-sum trees such that $L_{\mathcal{T}}'$ is obtained from $L_T$ by a sequence of pair direction reversals. We say that $\mathcal{T}$ and $\mathcal{T}'$ are \textit{equivalent}. Indeed, it is easy to see that $G(\mathcal{T})$ and $G(\mathcal{T}')$ are the same.

If each element of $\mathcal{G}$ is 2-connected, then an edge-sum tree $\mathcal{T} = (\mathcal{G}, T)$ is a \textit{block tree}. The next important kind of edge-sum tree, namely 3-block tree, due to Tutte, requires the following terminology. A \textit{3-block} is a simple 3-connected graph, a cycle with at least 3 edges, or a multi-edge with at least 3 edges. A \textit{3-block tree} is an edge-sum tree $\mathcal{T} = (\mathcal{G}, T)$ such that each element of $\mathcal{G}$ is a 3-block and such that if $\xi_i \xi_j \in E(T)$, then $G_i$ and $G_j$ are not both cycles and not both multi-edges.

Obviously, a 3-block tree is a block tree. Let us note that the edge-sum tree $\mathcal{T}$ that we saw in Figure 4 is a block tree, but not a 3-block tree. It follows easily
from the above proposition and Lemma 4.1 that composing a block tree produces a unique (unlabeled) 2-connected graph. It is natural to ask whether every 2-connected graph has a decomposition into some kind of block tree. Indeed, Tutte [7] proved the following:

4.2. Theorem. If $G$ is a 2-connected graph containing at least three edges, then it can be decomposed into a 3-block tree. Moreover, this decomposition is unique (up to equivalence of 3-block trees).

Later, we shall use the existence of such a decomposition guaranteed by Theorem 4.2.

For brevity, let us speak of the 3-block tree of a 2-connected graph $G$ rather than the class of equivalent 3-block trees of $G$. Next, we shall prove a useful lemma regarding the composition of a special kind of restriction of an edge-sum tree.

4.3. Lemma. If $\mathcal{T} = (\mathcal{G}, T)$ is an edge-sum tree and $\mathcal{T}' = (\mathcal{G}', T')$ is a restriction of $\mathcal{T}$ so that, for each node $\xi_j$ in $V(T) - V(T')$, the corresponding node graph $G_j$ is 2-connected, then $G(\mathcal{T}') \leq_{m} G(\mathcal{T})$.

Proof. We show that the hypotheses imply a stronger conclusion, namely $G(\mathcal{T}') \leq_{t} G(\mathcal{T})$. We may assume that $\mathcal{T}' = \mathcal{T} - \xi_j$, where $\xi_j$ is a leaf of $T$ whose corresponding node graph $G_j$ is 2-connected, since any subtree $T'$ can be obtained from $T$ by deleting leaves and since the taking of restrictions of edge-sum trees and the $\leq_{t}$ relation on graphs are transitive. Let $\varepsilon = \xi_j$ denote the link of $T$ incident with $\xi_j$. Then $(\{H, G_j\}, \varepsilon)$ is a partial composition of $\mathcal{T}$, where $H$, viewed as an unlabeled graph, is isomorphic to $G(\mathcal{T}')$. In this partial composition, each of $H$ and $G_j$
has an edge, respectively, $h$ and $g$, both of which are labeled $e$. Since $G_j$ is 2-connected, there is a cycle $C$ of length at least 2 that contains $g$. It follows that $H \oplus_2 C \leq_s H \oplus_2 G_j = G(T)$. Note that $H \oplus_2 C$ is isomorphic to the unlabeled graph obtained from $H$ by subdividing $h$ with $|C| - 2$ new vertices. Hence, $H \leq_{s} H \oplus_2 C$. Since $H \cong G(T')$, it follows that $G(T') \leq_{s} G(T')$.

The following is an immediate and useful corollary of Lemma 4.3.

4.4. Corollary. Let $G$ be a 2-connected graph, and let $T = (S, T)$ be its 3-block tree. If some element of $S$ is a 3-connected graph that has a cycle of length at least $N$, where $N$ is the number from Theorem 3.7, then $W_n \leq_m G$ (and hence $F_n \leq_m G$).

We shall need the following two well-known binary relations on graphs, which are more permissive versions of isomorphism. A graph $G$ is 2-isomorphic to a graph $H$, denoted $G \cong_2 H$, if there is a positive integer $n$ and a sequence $(G_i)_{i=1}^n$ of graphs such that $G_1 = G$, the final graph $G_n = H$, and if $i \in [n-1]_+$, then $G_{i+1}$ is obtained by performing one of the three following operations on $G_i$.

(i) Vertex identification: If $v_1$ and $v_2$ are vertices in distinct components of $G_i$, then $G_{i+1}$ is obtained by identifying $v_1$ and $v_2$ to a new vertex $v$.

(ii) Vertex cleaning: If $G^1$ and $G^2$ are disjoint graphs such that $G_i$ can be obtained from $G^1$ and $G^2$ by identifying a vertex $v_1$ of $G^1$ and a vertex $v_2$ of $G^2$ to a single vertex $v$, then let $G_{i+1} = G^1 \cup G^2$.

(iii) Twisting: Assume that $G^1$ and $G^2$ are disjoint graphs and that $u_1$, $u_2$, $v_1$, and $v_2$ are all distinct vertices with $\{u_1, v_1\} \subseteq V(G^1)$ and $\{u_2, v_2\} \subseteq V(G^2)$. Further, assume that $G_i$ is obtained from $G^1$ and $G^2$ by identifying $u_1$ and $u_2$ to a single vertex $u$ and by identifying $v_1$ and $v_2$ to a single vertex $v$. Call $G_{i+1}$ a twisting of $G_i$ about $\{u, v\}$ if $G_{i+1}$ is obtained from $G^1$ and $G^2$ by identifying $u_1$ and $v_2$ to a single vertex $u'$, and by identifying $u_2$ and $v_1$ to a single vertex $v'$.

If, in the process of obtaining $H$ from $G$, only the operations (i) and (ii) are used, we say that $G$ is 1-isomorphic to $H$ and we write $G \cong_1 H$. Note that if $H$ can be obtained from $G$ by adding isolated vertices, then $G \cong_1 H$.

The following lemma is straightforward—we leave its proof to the reader.

4.5. Lemma. If $T = (S, L, T)$ is an edge-sum tree, and $T' = (S, L', T)$ is an edge-sum tree obtained from $T$ by reversing the directions of some of the labeled edges of elements of $T$, then $G(T)$ is 2-isomorphic to $G(T')$.

Let $T = (S, L, T)$ be an edge-sum tree for which $|S| > 1$, let $H$ be a specified node graph in $S$, and let $\xi$ be the node that corresponds to $H$. Let the positive integer $m$ denote the number of links in $T$ incident with $\xi$, and let $\{\varepsilon_i : i \in [m]_+\}$ denote the set of links adjacent to $\xi$ in $T$. Then the star of $T$ (at $H$), denoted $T_\ast$, is the partial composition $T/(E(T) - \{\varepsilon_i : i \in [m]_+\})$ of $T$. We now define some additional notation regarding $T$ and $T_\ast$. For each $i \in [m]_+$, let $h_i$ be the edge of $H$ that is labeled $\varepsilon_i$, let $\xi_i$ be the endnode of $\varepsilon_i$ in $T$ that is not $\xi$, let $H_i$ be the node graph of $T$ corresponding to $\xi_i$, and let $k_i$ be the edge of $H_i$ that is labeled $\varepsilon_i$. Let $T_i$ be the restriction of $T_\ast$ induced by the component $T_i$ of $T \setminus \varepsilon_i$, containing $\xi_i$, and let $K^i = G(T_i)$.

It is straightforward that the set of node graphs of $T_\ast$ is $\{H\} \cup \{K^i : i \in [m]_+\}$, where $H$ is labeled as it is in $T$, and where $K^i$ has exactly one labeled edge, namely $k_i$, for each $i \in [m]_+$. 


The next lemma states that the operations of edge deletion and edge contraction commute with the process of forming the edge-sum tree. The proof is straightforward and its details are left for the reader.

4.6. Lemma. Let \( T = (G, L_G, T) \) be an edge-sum tree, let \( D \) and \( C \) be disjoint subsets of \( E(G(T)) \), and let \( T \setminus D \setminus C \) denote the edge-sum tree obtained by replacing each node graph \( H \in G \) with \( H' = H \setminus (E(H) \cap D)/(E(H) \cap C) \), and by replacing each directed labeling \( L_H \) in \( L_G \) with the directed labeling \( L_{H'} \) of \( H' \) induced by \( L_H \). Then \( G(T \setminus D \setminus C) = G(T) \setminus D \setminus C \). \( \square \)

5. A LONG PATH IN A 3-BLOCK TREE

The following is the main result of this section.

5.1. Theorem. Let \( G \) be a 2-connected graph with at least three edges, and let \( T = (G, T) \) be its 3-block tree. If \( n \) is a positive integer, and \( T \) contains a path of length at least \( 4(n - 1) + 1 \) as a subgraph, then \( F_n \leq_m G \).

Before proving Theorem 5.1, we need an auxiliary result, which is stated as Corollary 5.3 below. We begin by stating a result of Seymour [3].

5.2. Theorem. If \( M \) is a 3-connected matroid that has a minor in the set \( \mathcal{F} = \{U_{2,4}, M(K_4)\} \), and \( X \) is any subset of \( E(M) \) that has at most two elements, then \( M \) has a minor in \( \mathcal{F} \) using \( X \). \( \square \)

Two well-known facts: that every 3-connected graph contains a minor isomorphic to \( K_4 \) and that the matroid \( U_{2,4} \) is not graphic, together with Theorem 5.2 immediately imply the following:

5.3. Corollary. If \( G \) is a simple 3-connected graph, and \( e \) and \( f \) are edges of \( G \), then there is a \( K_4 \)-minor of \( G \) that uses \( e \) and \( f \). \( \square \)

The remainder of this section is devoted to proving Theorem 5.1.

Proof of Theorem 5.1. Assume that \( T \) contains a subtree \( P_0 \) that is a path of length at least \( 4(n - 1) + 1 \). If each of the elements of \( G \) corresponding to the endnodes of \( P_0 \) is a multi-edge, then let \( P \) be a subpath of \( P_0 \) obtained by deleting an endnode from \( P_0 \); otherwise let \( P = P_0 \). Then \( T \) contains a subpath \( P \) of length \( N \), for some integer \( N \geq 4(n - 1) \), one endnode of which corresponds to a 3-connected graph or a cycle. Let \( T' = (G', P) \) be the 3-block tree that is the restriction of \( T \) induced by \( P \), and let \( G' = G(T') \). By Lemma 4.3, \( G' \leq_m G \).

By renumbering indices, we may assume that the node set of \( P \) is \( \{\xi_i : i \in [N]\} \), the link set of \( P \) is \( \{\xi_{i-1}, \xi_i : i \in [N]\} \), and \( G' = \{G_i : i \in [N]\} \), where \( G_i \) is the 3-block corresponding to \( \xi_i \), and \( G_N \) is not a multi-edge. We now want to partition \( G' \) into singletons and pairs as follows. Let \( i \) be the largest index such that \( G_i \) does not belong to a singleton or a pair of elements of \( G' \). If \( G_{i-1} \) is a multi-edge, then form the pair \( \{G_{i-1}, G_i\} \); otherwise, form the singleton \( \{G_i\} \). If all elements of \( G' \) have not been placed in a singleton or a pair, then repeat this process. It is straightforward that this process produces a partition \( \mathcal{P}(G') \) of \( G' \) where each element of \( \mathcal{P}(G') \) is a singleton consisting of a cycle or a 3-connected graph, or a pair \( \{G_{i-1}, G_i\} \) consisting of a multi-edge \( G_{i-1} \) and a 3-block \( G_i \) that is not a multi-edge. Since each element of \( \mathcal{P}(G') \) consists of at most two 3-blocks, \( |\mathcal{P}(G')| = N' + 1 \) for some integer \( N' \geq \left[\frac{N+1}{2}\right] - 1 \). Note that it follows from the
way that \( \mathcal{P}(\mathcal{S}') \) is defined and from the fact that \( \mathcal{S}' \) is a 3-block tree that if \( i \in [N]_+ \), and \( G_i \) is a cycle that makes up a singleton in \( \mathcal{P}(\mathcal{S}') \), then \( G_{i-1} \) is 3-connected.

Note that \( E(P) \) is partitioned into sets \( E' \) and \( E'' \) of links such that if \( e' \in E' \), then the node graphs that contain an edge labeled \( e' \) are contained in different elements of \( \mathcal{P}(\mathcal{S}') \), and if \( e'' \in E'' \), then the node graphs that contain an edge labeled \( e'' \) form a pair in \( \mathcal{P}(\mathcal{S}') \).

Let \( T'' \) be the block tree obtained by contracting \( E'' \) in \( T' \). It follows that \( P'' = P/E'' \) is the tree of \( T'' \), and \( \mathcal{S}'' = \{ G_i : \{ G_i \} \in \mathcal{P}(\mathcal{S}') \} \cup \{ G_i \oplus G_{i+1} : \{ G_i, G_{i+1} \} \in \mathcal{P}(\mathcal{S}') \} \) is the set of node graphs of \( T'' \). Furthermore, \( |\mathcal{S}'| = |\mathcal{P}(\mathcal{S}')| = N' + 1 \). Also, it is evident that \( G(T'') = G(T') = G' \) since \( T'' \) is a partial composition of \( T' \). Let \( G_0 \) denote the element of \( \mathcal{S}'' \) that is either \( G_N \) or \( G_{N-1} \oplus G_N \), let \( e'_i \) denote the endnode of \( P' \) corresponding to \( G_0' \), and if \( E(P') \) is non-empty, then let \( e'_i \) denote the link of \( P' \) incident with \( \xi_0 \). If \( P' \) contains additional links, then let \( \{ e''_i \}_{i=1}^N \) denote the remaining links of \( P' \) such that \( e'_i \) and \( e''_{i+1} \) are adjacent for each \( i \in [N - 1]_+ \), and rename the nodes of \( P' \) and elements of \( \mathcal{S}'' \) such that, for each \( i \in [N]_+ \), the endnodes of the link \( e'_i \) are \( \xi'_i \) and \( \xi''_i \), and \( G'_i \) is the element of \( \mathcal{S}'' \) corresponding to \( \xi'_i \). It follows that if \( G'_i \) is a cycle, then \( G'_i \) is a 3-connected graph.

Note that each element of \( \mathcal{S}'' \) is a 3-block that is not a multi-edge, or it is the 2-sum of a 3-block that is a multi-edge and a 3-block that is not a multi-edge. It follows that all edges of \( G_0' \) are unlabeled except for one edge \( f_0 \) that is labeled \( e'_1 \), and all edges of \( G_1' \) are unlabeled except for one edge \( e_{N-1} \) that is labeled \( e''_{N-1} \). Furthermore, if \( G_0' \) is not a simple graph, then \( f_0 \) is contained in the proper multi-edge of \( G_0' \). Also, if \( G_1' \) is not a simple graph, then \( e_{N-1} \) is a trivial multi-edge of \( G_1' \). If \( i \in [N - 1]_+ \), then all edges of \( G_i' \) are unlabeled except for an edge \( e_i \) that is labeled \( e'_i \) and an edge \( f_i \) that is labeled \( e''_{i+1} \). Moreover, if \( G_i' \) is not simple, then \( e_i \) is a trivial multi-edge of \( G_i' \), and \( f_i \) belongs to the proper multi-edge of \( G_i' \). Let \( e_0 \) be an unlabeled trivial multi-edge of \( G_0' \), and let \( f_N' \) be an edge that is contained in a largest unlabeled multi-edge of \( G_N' \), so that each element \( G_i' \) of \( \mathcal{S}'' \) has exactly two specified edges \( e_i \) and \( f_i \).

![Diagram](image)

**Figure 5.** The graphs \( D_i, D'_i, \) and \( D''_i \).

We shall show that, for each \( i \in [N] \), the graph \( G_i' \) contains a particular minor isomorphic to one of the three graphs in Figure 5. First, we shall show that if \( G_i' \) is a 3-block that is a cycle, then \( D_i \leq_m G_i' \). Then, we shall show that if \( G_i' \) is the 2-sum of a 3-block that is a cycle and a 3-block that is a multi-edge, then \( D'_i \leq_m G_i' \). For the remaining case, in which \( G_i' \) is a 3-connected graph, we shall show that \( D''_i \leq_m G_i' \).

First, assume that \( G_i' \) is a 3-block that is a cycle. Since \( G_i' \) has at least three edges, \( G_i' \setminus \{e_i, f_i\} \) consists of a proper path \( P_1 \) and a (perhaps trivial) path \( P_2 \). By
contracting, in \( G_i' \), the paths \( P_1 \) to a single edge and \( P_2 \) to a vertex, we obtain a graph \( G_i'' \) that is isomorphic to \( D_j' \).

Now, assume that \( G_i' \) is the 2-sum of a 3-block \( C \) that is a cycle and a 3-block \( C^* \) that is a multi-edge. Clearly, the simplification of \( G_i' \) is a cycle with at least three edges. As already mentioned, \( e_i \) is a trivial multi-edge of \( G_i' \), and \( f_i \) is contained in the proper multi-edge of \( G_i' \). As in the case in which \( G_i' \) is a cycle, the graph obtained by deleting the proper multi-edge and \( e_i \) from \( G_i'' \) consists of a proper path \( P_1 \) and a (perhaps trivial) path \( P_2 \). If we contract, in \( G_i'' \), the paths \( P_1 \) to a single edge and \( P_2 \) to a vertex, then the resulting graph contains a subgraph \( G_i''' \) that is isomorphic to \( D_j'' \).

Finally, assume that \( G_i'' \) is a 3-connected graph. If \( G_i'' \) is not simple, then \( e_i \) is a trivial multi-edge, and \( f_i \) is contained in the proper multi-edge of \( G_i'' \), and hence, \( e_i \) and \( f_i \) are not parallel. Consequently, if \( G_i'' \) is not simple, then we may take the simplification \( \overline{G_i} \) of \( G_i' \) so that \( \{e_i, f_i\} \subseteq E(\overline{G_i}) \). By Corollary 5.3, \( \overline{G_i} \) has a \( K_4 \)-minor using \( e_i \) and \( f_i \). Thus, one of the two graphs in Figure 6 is a minor of \( \overline{G_i} \), and, by contracting the shaded edge in either graph, we obtain a graph \( G_i''' \) that is isomorphic to \( D_j''' \). Since \( \overline{G_i} \) is a graph that is, in some sense, similar to a fan.

So far, we have been disregarding the directions assigned by the directed labeling of \( \mathcal{T}'' \) to the labeled edges in the elements of \( \mathcal{S}'' \). We consider these directions now. By performing the appropriate pair direction reversals, we may assume that, for each \( i \in [N'] \), the edge \( f_i \) of \( G_i'' \) is directed so that its head is incident with \( e_i \). Let \( \mathcal{T}' \) denote the block tree obtained from \( \mathcal{T}'' \) by directing \( e_i \in E(G_i'') \) so that its head is incident with \( f_i \), for each \( i \in [N'] \). Since the simplification of each \( G_i'' \) is a triangle (that is, a 3-cycle) for each \( i \in [N'] \), call the vertex common to \( e_i \) and \( f_i \) the point of \( G_i'' \), and let \( g_i \) denote the edge of \( G_i'' \) that is not adjacent to the point of \( G_i'' \).

We want to show that the simplification of \( G_i'' \) is 2-isomorphic to \( F_{N'+2} \). By Lemma 4.5 it suffices to show that the simplification of \( G(\mathcal{T}') \) is isomorphic to \( F_{N'+2} \). Informally, in the composition of \( G(\mathcal{T}') \), the first node graph \( G_0'' \) contributes 2 to the size of the fan, and each additional node graph \( G_i'' \) contributes 1 to the size of the fan. Let us recall that if \( G_{i-1}' \) is a cycle, then \( G_i' \) is a 3-connected

![Figure 6. One of the above graphs is a minor of \( G_i' \).](image-url)
graph. It follows that if \( G''_{i-1} \cong D_{i-1} \), then \( G''_{i} \cong D''_{i} \). It is straightforward that \( \widetilde{G}(T^*) \cong F_{N+2} \), given the way that the labeled edges of \( T^* \) are directed and the fact that if \( G''_{i-1} \cong D_{i-1} \), then \( G''_{i} \cong D''_{i} \). Hence, \( \widetilde{G}'' \cong F_{N+2} \)

Next, we want to show that the \( \left( \begin{pmatrix} N' \times 2 \end{pmatrix} \right) \)-fan is a minor of \( \widetilde{G}'' \). Since \( \widetilde{G}'' \) is 2-isomorphic to \( F_{N+2} \) and 2-connected, \( \widetilde{G}'' \) can be obtained from a finite sequence of twistings of \( F_{N+2} \) about vertex-cuts of size two. It is straightforward that \( \widetilde{G}'' \) is similar to a fan, where some of the triangles may point up and some may point down instead of all triangles pointing in the same direction.

Define the function \( f: \mathbb{S}'' \to \{-1, 1\} \) as follows. Let \( f(G''_i) = 1 \), and inductively, for each \( i \in [N]' \), if the directed labeling of \( \mathbb{S}'' \) directs \( e_i \) so that its head is the point of \( G''_i \), then \( f(G''_i) = f(G''_{i+1}) \); otherwise, \( f(G''_i) = -f(G''_{i+1}) \). Informally, we shall say that the triangle of \( \mathbb{S}'' \) with base \( y_i \) points up if \( f(G''_i) = 1 \) and points down if \( f(G''_i) = -1 \). It follows that if \( \sum_{i=0}^{N'} f(G''_i) \geq 0 \), then at least half of the triangles of \( \mathbb{S}'' \) point up; otherwise, more than half of the triangles point down. If \( \sum_{i=0}^{N'} f(G''_i) \geq 0 \), then contract \( \{y_i : f(G''_i) = -1\} \) in \( \mathbb{S}'' \); otherwise, contract \( \{y_i : f(G''_i) = 1\} \). It follows that the simplification of the resulting graph is isomorphic to a fan of size at least \( \left\lfloor \frac{N' + 2}{2} \right\rfloor \). Hence, \( F_{\left\lfloor \frac{N'}{2} \right\rfloor} \leq_m \mathbb{G}'' \leq_m G' \), and

\[
\left\lfloor \frac{N' + 2}{2} \right\rfloor \geq \left\lfloor \frac{\left\lfloor \frac{N'}{2} \right\rfloor + 1}{2} \right\rfloor \geq \left\lfloor \frac{\frac{n(n-1)+1}{2} + 1}{2} \right\rfloor = \frac{\frac{2(n-1) + 2}{2}}{2} = n.
\]

Thus, \( F_n \leq_m G \).

\[ \square \]

### 6. Cocompactness and Multipaths

In this section we state and prove a lemma which states that if a graph \( G \) satisfies certain conditions that depend, in part, on an integer \( n \) exceeding 3, then an element of \( \{C_{n,n}^s, P_{n,n} \} \) is a minor of \( G \), where \( P_{n,n} \) is obtained from the path \( P_n \) on \( n \) edges by replacing each edge of \( P_n \) with a multi-edge of size \( n \). Following the proof of the lemma, we state two corollaries, which describe the consequences of the lemma to 2-connected graphs and block trees.

#### 6.1. Lemma. Let \( G \) be a graph with two specified vertices \( x \) and \( y \) such that \( G \cup e \) is 2-connected, where \( e \parallel xy \), and let \( n \) be an integer exceeding 3. If every \( xy \)-path in \( G \) has length at least \( n(n-1) \) and every \( xy \)-edge-cut in \( G \) has size at least \( n2n^2 \), then at least one of the following holds.

(i) \( C_{n,n}^s \leq_m G \), and the edges of \( C_{n,n}^s \) that have degree \( n \) are \( x \) and \( y \).

(ii) \( P_{n,n} \leq_m G \), and the endvertices of \( P_{n,n} \) are \( x \) and \( y \).

**Proof.** Label each vertex \( v \) of \( G \) with its distance \( d(v) \) from \( x \). Then \( d(y) = N \) for some \( N \geq n(n-1) \). For \( i \in [N] \), let \( V_i \) be the set of those vertices \( v \) labeled with \( i \) such that there is a \( xy \)-path in \( G \) each of whose vertices, except \( v_i \), is labeled with an integer exceeding \( i \). It is clear that \( V_i \) is non-empty if \( i \in [N] \), that \( V_0 = \{x\} \) and \( V_N = \{y\} \), and that \( V_i \) is an \( xy \)-vertex-cut if \( i \in [N-1]_+ \). Let \( V^x \) be the set of vertices in the component of \( G - V_i \) containing \( x \) for \( i \in [N]_+ \), and let \( V^y \) be the set of vertices in the component of \( G - V_i \) containing \( y \) for \( i \in [N-1] \). We now establish some properties of these two sets.
(1) \( V_i \subseteq V^y_i \) if \( 0 \leq i' < i \leq N \).

To see this, it suffices to show that, for every \( v \in V_i \), there is an \( xy \)-path in \( G - V_i \). Let \( v \) be an arbitrarily chosen vertex in \( V_i \). Since the label on \( v \) is determined by its distance from \( x \), it follows that \( G \) contains an \( xy \)-path \( P_v \) of length \( l(v) = i' \) and that the label of each vertex of \( P_v \) is at most \( i' \). In particular, \( P_v \) has no vertex of \( V_i \), and thus \( P_v \) is contained in \( G - V_i \), as required.

(2) \( V_i \subseteq V^y_i \) if \( 0 \leq i < i' \leq N \).

The proof of (2) is very similar to the proof of (1). Let \( v \) be an element of \( V_i \). It follows that \( G \) contains a \( xy \)-path \( P_v \) and that the label of each vertex of \( P_v \) is at least \( i' \). In particular, \( P_v \) contains no vertex of \( V_i \), and thus \( P_v \) is contained in \( G - V_i \). Consequently, (2) holds.

For each \( i \) in \([N-n]\), define \( V^y_i = V^y_i \cap V^x_{i+n} \). Statements (1) and (2) immediately imply that for each such \( i \), \( V_{i+i} \subseteq V^y_i \), and hence \( V^y_i \) contains an \( xy \)-vertex-cut.

Assume first that there is an \( i \in [N-n] \) such that a smallest \( xy \)-vertex-cut \( S_i \) of \( G \) contained in \( V^y_i \) has at least \( n \) vertices. Since \( G \) is connected, it is clear that there are three kinds of bridges of \( V_i \cup V_{i+n} \) in \( G \): those that meet only \( V_i \), those that meet only \( V_{i+n} \), and those that meet both \( V_i \) and \( V_{i+n} \).

Now, we want to contract all of the bridges of \( V_i \cup V_{i+n} \) except those that meet both \( V_i \) and \( V_{i+n} \). More precisely, let us contract \( E_0 = E(G) - (E(G[V^y_i]) \cup E(V_i, V^y_i) \cup E(V^y_i, V_{i+n})) \) in \( G \), where \( E(X_1, X_2) \) denotes the set of edges whose elements have one vertex in \( X_1 \) and the other vertex in \( X_2 \) for disjoint sets \( X_1 \) and \( X_2 \) of vertices. Let \( G_0 = G/E_0 \). On contracting \( E_0 \) in \( G \), it is easy to see that \( x \) and the vertices of \( V_i \) are identified, and that \( y \) and the vertices of \( V_{i+n} \) are identified. It is natural to let \( x \) and \( y \), respectively, denote these vertex identifications. Consequently, \( V(G_0) = V^y_i \cup \{x, y\} \).

The next part of the proof uses the following two simple observations. First, no edge of \( E_0 \) has a vertex in \( V^y_i \). Second, two vertices are in the same component of a graph if and only if after contracting any set of edges, those vertices (which may become identified) are in the same component of the resulting graph.

We now show that \( S_i \) is a smallest \( xy \)-vertex-cut of \( G_0 \) by showing that \( S_i \) contains an \( xy \)-vertex-cut of \( G_0 \), and then showing that no subset of \( V(G_0) - \{x, y\} \) having size less than \( |S_i| \) is an \( xy \)-vertex-cut of \( G_0 \). Clearly \( x \) and \( y \) are in different components of \( G - S_i \) since \( S_i \) is an \( xy \)-vertex-cut of \( G \). Also, in view of the first observation above, \((G - S_i)/E_0 \) is well-defined since \( S_i \subseteq V^y_i \). Thus, \((G - S_i)/E_0 = (G/E_0) - S_i = G_0 - S_i \), and hence \( x \) and \( y \) are in different components of \( G_0 - S_i \) by the second observation above. Now let \( S \) be any subset of \( V^y_i \) that has fewer than \( |S_i| \) vertices. Then \( x \) and \( y \) are in the same component of \( G - S \) since \( S_i \) is a smallest \( xy \)-vertex-cut of \( G \) contained in \( V^y_i \). By the first observation above, \((G - S)/E_0 \) is well-defined since \( S \subseteq V^y_i \), and thus \((G - S)/E_0 = G_0 - S \). By the second observation above, \( x \) and \( y \) are in the same component of \( G_0 - S \). Hence, no subset of \( V^y_i \) that has fewer than \( |S_i| \) vertices is an \( xy \)-vertex-cut of \( G_0 \). It follows that \( S_i \) is a smallest \( xy \)-vertex-cut of \( G_0 \). Since \( |S_i| \geq n \), (1) implies that there are \( xy \)-paths \( P_1, P_2, \ldots, P_n \) in \( G_0 \) that are pairwise internally vertex-disjoint.

Now, we show that each \( P_j \) has length at least \( n \) for \( j \in [n]_+ \). It follows from the first observation above that \( G[V^y_i] = G_0[V^y_i] \). Let \( P \) be any \( xy \)-path in \( G_0 \). Then \( P' = P - \{x, y\} \) is a path in \( G_0[V^y_i] = G[V^y_i] \). Hence, \( P' \) is a path in \( G \) that has one endvertex adjacent to some vertex of \( V_i \) and the other endvertex adjacent to some vertex of \( V_{i+n} \). It is clear that if two vertices are adjacent in \( G \), then their
labels differ by 0 or 1. This implies that if the labels of the endvertices of a path in \( G \) are \( l_1 \) and \( l_2 \), then the length of that path is at least \( |l_2 - l_1| \). Furthermore, one endvertex of \( P' \) is labeled at most \( i + 1 \), and the other endvertex is labeled at least \( i + n - 1 \). So the length of \( P' \) is at least \((i + n - 1) - (i + 1) = n - 2\). Hence, the length of \( P \) in \( G_0 \) is at least \( n \). In particular, \( P_j \) has length at least \( n \), for each \( j \in [n]_+ \).

Let \( G'_0 \) be the subgraph of \( G_0 \) that is the union of \( P_1, P_2, \ldots, P_n \). Then \( G'_0 \) consists of \( n \) pairwise internally vertex-disjoint \( xy \)-paths, all of length at least \( n \). On contracting an appropriate number of interior edges of \( P_j \) in \( G'_0 \), for each \( j \in [n]_+ \), we obtain a minor of \( G_j \) that is isomorphic to \( C_{n,n} \), whose vertices of degree \( n \) are \( x \) and \( y \). So \( C_{n,n} \leq_m G'_0 \leq_s G_0 \leq_m G \) (hence, \( C_{n,n} \leq_m G \)), and the vertices of degree \( n \) of \( C_{n,n} \) are \( x \) and \( y \). Thus, the lemma holds if there is an \( i \in [N - n] \) such that \( V_i \) lacks an \( xy \)-vertex-cut of size less than \( n \).

Now, for the remaining case, assume that, for each \( i \in [N - n] \), if \( S_i \) is a smallest \( xy \)-vertex-cut in \( V_i \), then \( |S_i| < n \). Let \( S'_i \) be a smallest \( xy \)-vertex-cut from \( V_i \) for each \( i \in [n - 2] \). As \( S'_i \) is an \( xy \)-vertex-cut, each \( xy \)-path must pass through some vertex \( s_i \) in \( S'_i \), for each \( i \in [n - 2] \).

Let us consider the bridges of \( \bigcup_{i=0}^{n-2} S'_i \) in \( G \). Since \( G \) is connected, possibly, we could have the following kinds of bridges: those that meet exactly one \( S'_i \), those that meet only \( S'_i \) and \( S'_{i+1} \) for some \( i \in [n - 3] \), and those that meet \( S'_i \) and \( S'_j \) (and, perhaps, additional sets \( S'_k \)) for some \( 0 \leq i < j - 1 < n - 2 \). Next, we show that \( G \) has no bridges of this last kind by showing that any \( s_is_j \)-path in \( G \) contains a vertex \( s_{i+1} \in S'_{i+1} \), when \( 0 \leq i < j - 1 < n - 2 \), \( s_i \in S'_i \), and \( s_j \in S'_j \).

First, we point out that if the labels of the endvertices of a path in \( G \) are \( l_1 \) and \( l_2 \), then certainly the path has at least one vertex labeled \( l' \) for each integer \( l' \) between \( l_1 \) and \( l_2 \), since the labels of adjacent vertices in \( G \) differ by 0 or 1. It follows that if \( P \) is an \( s_is_j \)-path in \( G \), where \( 0 \leq i < j - 1 < n - 2 \), and \( s_i \) and \( s_j \) are arbitrary elements of \( S'_i \) and \( S'_j \), respectively, then \( P \) contains a vertex whose label is \((i + 1)n \) since \( l(s_i) = in \) and \( l(s_j) = jn \). Let \( s_{i+1} \) be the vertex labeled \((i + 1)n \) that is closest in \( P \) to \( s_j \). Then each vertex of the \( s_is_{i+1}s_j \)-subpath of \( P \), except \( s_{i+1} \), is labeled greater than \((i + 1)n \). Since \( s_i \in S'_i \), there is an \( s_is_j \)-path each of whose vertices is labeled at least \( in \). The union of the \( s_is_{i+1}s_j \)-subpath and the \( s_is_j \)-path contains an \( s_{i+1}s_j \)-path each of whose vertices is labeled greater than \((i + 1)n \), except \( s_{i+1} \), which is labeled \((i + 1)n \). Hence, \( s_{i+1} \in S'_{i+1} \), which establishes that \( G \) has no bridges that meet \( S'_i \) and \( S'_j \), where \( 0 \leq i < j - 1 < n - 2 \). Consequently, the structure of the bridges of \( \bigcup_{i=0}^{n-2} S'_i \) in \( G \) is as in Figure 7.

Now, let us consider the minor \( G_1 \) of \( G \) that is obtained by contracting those bridges of \( \bigcup_{i=0}^{n-2} S'_i \) in \( G \) that contain neither \( x \) nor \( y \) and that meet only \( S'_i \), for each \( i \in [n - 2] \). These bridges are represented by the shaded portions of \( G \) in Figure 7. We note that, for each \( i \in [n - 2] \), some vertices of \( S'_i \) may become identified on contracting \( G \) to \( G_1 \); let \( S'_{i,1} \) denote the subset of \( V(G_1) \) that corresponds to \( S'_i \in V(G) \). It is clear that \( |S'_{i,1}| \leq |S'_i| \).

We now consider the minor \( G_2 \) of \( G_1 \) that is obtained by contracting the edge set \( E_1 \), contained in \( G_1 \), that is defined as follows, \( E_1 = E(G_1) - (E(S'_1); y) \cup \bigcup_{i=0}^{n-2} E(S'_i; y)) \), where \( E(S'; y) \) is the set of those edges of \( G_1 \) each of which has one vertex in \( S \) and the other vertex in the component of \( G_1 - S \) containing \( y \), for \( S \subseteq V(G_1) - y \). For each \( i \in [n - 2] \), some vertices of \( S'_{i,1} \) may become identified
on contracting $G_1$ to $G_2$: let $S^1_{i,2}$ denote the set of vertices of $G_2$ that corresponds to $S^0_{i,1}$ in $G_1$. Then $|S^1_{i,2}| \leq |S^0_{i,1}|$, and $G_2 = G_1 / E_1$. Figure 8 shows a typical $G_2$.

We now show that $G_2$ has at least $n2^n$ pairwise edge-disjoint $xy$-paths. Recall that every $xy$-edge-cut of $G$ has size at least $n2^n$. Then by the well-known Menger's Theorem, $G$ has at least $n2^n$ pairwise edge-disjoint $xy$-paths. Note that, given any $xy$-path $P$ of $G$, if we contract (in $G$) a set $S$ of edges that contains no $xy$-path, then the subgraph $P'$ of $G / S$ induced by $E(P) - S$ is connected, and hence $P'$ contains an $xy$-path $P''$. Moreover, $E(P'') \subseteq E(P') \subseteq E(P)$. This containment and the fact that $G$ has at least $n2^n$ pairwise edge-disjoint $xy$-paths imply that $G_2$ has at least $n2^n$ pairwise edge-disjoint $xy$-paths. Next, we show that the simplification of $G_2$ has fewer than $2^n$ $xy$-paths.

Note that each edge of $G_2$ is of the form $xS_0$, $s_n-2y$, or $s_is_{i+1}$, where $s_0 \in S^0_{0,2}$, $s_{n-2} \in S^0_{n-2,2}$, and $s_i \in S^0_{i,2}$, for $i \in [n-3]$. It follows that each $xy$-path in $G_2$ has
length at least \( n \). Moreover, the simplification \( \widetilde{G}_2 \) of \( G_2 \) has at most \(|S_{0,2}^1| \) edges between \( x \) and \( S_{0,2}^1 \), at most \(|S_{1,2}^1| \) edges between \( S_{1,2}^1 \) and \( S_{1,2}^1 \), and at most \(|S_{n-2,2}^1| \) edges between \( S_{n-2,2}^1 \) and \( y \). Since \(|S_{1,2}^1| \leq |S| < n \), for each \( i \in [n-2] \), \( \widetilde{G}_2 \) has at most \( n-1 + (n-2)(n-1)^2 + n-1 \) edges, and hence \( \widetilde{G}_2 \) has fewer than \( n^2 \) edges. Clearly, the collection of \( xy \)-paths in \( \widetilde{G}_2 \) is contained in the collection \( S \) of subgraphs of \( \widetilde{G}_2 \) that lack isolated vertices. Since \(|S| < 2^n \), there are fewer than \( 2^n \) \( xy \)-paths in \( \widetilde{G}_2 \).

Since \( G_2 \) has at least \( n \) pairwise edge-disjoint \( xy \)-paths and \( \widetilde{G}_2 \) has fewer than \( 2^n \) \( xy \)-paths, there are at least \( n \) pairwise edge-disjoint \( xy \)-paths, \( P_1', P_2', \ldots, P_n' \) in \( G_2 \), each of length at least \( n \), that use the same vertices in the same order. If the length of \( P_j' \) is greater than \( n \) for each \( j \in [n] \), then we can contract in \( \bigcup_{j=1}^n P_j' \) a parallel class whose edges are incident to neither \( x \) nor \( y \) repeatedly until we obtain a graph isomorphic to \( P_{n,n} \) whose endvertices are \( x \) and \( y \). So \( P_{n,n} \leq_m \bigcup_{j=1}^n P_j' \leq_m G_2 \leq_m G_1 \leq_m G \) (hence, \( P_{n,n} \leq_m G \)), and the endvertices of \( P_{n,n} \) are \( x \) and \( y \). Thus, the lemma holds.

Now, we shall describe how Lemma 6.1 can be applied to 2-connected graphs and block trees. The application to 2-connected graphs, stated in Corollary 6.2 below, is more intuitive and requires less notation than the application to block trees in Corollary 6.3 that follows it.

**6.2. Corollary.** Let \( B \) be a bridge of \( \{x, y\} \) in a 2-connected graph \( G \), for distinct vertices \( x \) and \( y \) in \( G \). If each \( xy \)-path in \( B \) has length at least \( n(n-1) \), and if each \( xy \)-edge-cut in \( B \) has size at least \( n \), then an element of \( \{C_{n,n}, C_{n,n}'\} \) is a minor of \( G \).

We omit the proof of Corollary 6.2 because it is very similar to the proof of Corollary 6.3, which is presented below, and to prove Corollary 6.2 would require the introduction of a large amount of notation, as in the statement of Corollary 6.3. It will be straightforward, once Corollary 6.3 is proved, that Corollary 6.3 is, in some sense, a special case of Corollary 6.2. Now, we state and prove Corollary 6.3.

**6.3. Corollary.** Let \( T = (S, T) \) be a block tree. For each link \( \varepsilon \) of \( T \), consider the partial composition \( T_{\varepsilon} = ((H_{\varepsilon}, H_{\varepsilon}'), (E(T) - \varepsilon)) \) of \( T \), and, for each \( i \in \{1, 2\} \), let \( h_{\varepsilon}^i \parallel u_{\varepsilon}^i v_{\varepsilon}^i \) denote the edge of \( H_{\varepsilon}^i \) labeled \( \varepsilon \). If there are a link \( \varepsilon \in E(T) \), an index \( i \in \{1, 2\} \), and an integer \( n \) exceeding 3, such that each \( u_{\varepsilon}^i v_{\varepsilon}^i \)-path in \( H_{\varepsilon}^i \) has length at least \( n(n-1) \) and each \( u_{\varepsilon}^i v_{\varepsilon}^i \)-edge-cut in \( H_{\varepsilon}^i \) has size at least \( n^2 \), then \( C_{n,n} \leq_m G(T) \) or \( C_{n,n-2} \leq_m G(T) \).

**Proof.** Assume that the link \( \varepsilon \) of \( T \) and the integers \( i \) and \( n \) satisfy the hypotheses. Since \( T \) is a block tree, \( H_{\varepsilon}^i \) is 2-connected. By Lemma 6.1, either \( C_{n,n} \leq_m H_{\varepsilon}^i \) or \( P_{n,n} \leq_m H_{\varepsilon}^i \) and the endvertices of \( P_{n,n} \) are \( u_{\varepsilon}^i \) and \( v_{\varepsilon}^i \). Since \( H_{\varepsilon}^i \) is the composition of one of the restrictions of \( T \) induced by one of the components of \( T \), it follows from Corollary 6.2 that \( H_{\varepsilon}^i \leq_m G(T) \). Consequently, \( C_{n,n} \leq_m G(T) \) or \( P_{n,n} \leq_m G(T) \). Since \( P_{n,n} \) and \( h_{\varepsilon}^i \) each have \( u_{\varepsilon}^i \) and \( v_{\varepsilon}^i \) as endvertices, \( (\ v_{\varepsilon}^i \cup h_{\varepsilon}^i ) \leq_m C_{n,n} \). The result follows.

7. \( n \)-Close Block Trees

In this section, we shall concentrate on graphs that do not satisfy the hypotheses of Corollary 6.3. We formalize this is follows. Let \( n \) be an integer exceeding three,
and let $\mathcal{T} = (S, T)$ be a block tree. For each link $\varepsilon$ of $T$, let $\mathcal{T}_\varepsilon$ denote the partial composition $\langle \{H^1_\varepsilon, H^2_\varepsilon\}, T/(E(T) - \varepsilon) \rangle$ of $\mathcal{T}$, and, for each $i \in \{1, 2\}$, let $h^i_\varepsilon$ denote the edge of $H^i_\varepsilon$ labeled $\varepsilon$. We call $\mathcal{T}$ an $n$-close block tree if for every link $\varepsilon$ of $T$ and each $i \in \{1, 2\}$ at least one of the following holds.

(i) Every $u^i_\varepsilon v^i_\varepsilon$-path in $H^i_\varepsilon \backslash h^i_\varepsilon$ has length less than $n(n - 1)$.
(ii) Every $u^i_\varepsilon v^i_\varepsilon$-edge-cut in $H^i_\varepsilon \backslash h^i_\varepsilon$ has size less than $n2^n$.

We have already seen in Corollary 6.3 that if the 3-block tree of a 2-connected graph has a 3-connected node graph with a cycle of length at least $N$, where $N$ is the number from Theorem 3.7 that depends on $n$, then $F_n$ is a minor of $G$. Also, we have seen in Theorem 5.1 that if the tree of the 3-block tree of $G$ contains a path of length at least $4(n - 1) + 1$, then $F_n$ is a minor of $G$. Additionally, we have seen in Corollary 4.4 that if the 3-block tree of $G$ is not $n$-close, for some integer $n$ exceeding 3, then $C_{n,n}$ or $C_{n,n}^*$ is a minor of $G$. So, we may restrict our attention to an arbitrary $n$-close 3-block tree $\mathcal{T}$ whose tree has no path of length exceeding $4(n - 1)$ and whose 3-connected node graphs have no cycles of length exceeding $N$, where $n > 3$ and $N$ is the number from Theorem 3.7 depending on $n$. In this section, we shall show that if $\mathcal{T}$ is such a 3-block tree, then the type of $G(\mathcal{T})$ is bounded from above by a function of $n$, or $C_{n,n-2} \leq_m G(\mathcal{T})$, or $C_{n,n-2}^* \leq_m G(\mathcal{T})$.

Before we can state and prove any results in this section, we need to make some definitions and assumptions, and develop some terminology. By a rooted edge-sum tree we mean an edge-sum tree $\mathcal{T} = (S, T)$ whose tree $T$ is a rooted tree (that is, $T$ contains a distinguished node $\xi$ called the root of $T$). If $H$ is the node graph in $\mathcal{T}$ that corresponds to $\xi$, then call $H$ the root graph of $\mathcal{T}$. The depth of $T$, denoted $D(T)$, is $\max\{d_T(\xi, \eta) : \eta \in V(T)\}$, where $d_T(\xi, \eta)$ is the distance in $T$ between the root $\xi$ of $T$ and $\eta$. We will sometimes abuse terminology and notation by referring to the root and the depth of $\mathcal{T}$ rather than to the root and the depth of $T$.

It is easy to see that if $T$ has no path of length exceeding $2M$, where $M$ is a nonnegative integer, then, by distinguishing an appropriate vertex of $T$, the edge-sum tree $\mathcal{T}$ can be viewed as a rooted edge-sum tree of depth at most $M$.

As noted earlier, we may restrict our attention to an arbitrary $n$-close 3-block tree whose tree has no path of length exceeding $4(n - 1)$ and whose 3-connected node graphs have no cycles of length exceeding $N_n = N$, where $n$ is an integer exceeding 3 and $N$ is the number form Theorem 3.7 depending on $n$. Clearly, if we think of such a 3-block tree as being rooted, then we may view it as having depth at most $M_n = 2(n - 1)$. We shall see that these values $M_n$ and $N_n$, that depend only on an integer $n$ that exceeds 3, appear in several of the results of this section.

Let $n$ be an integer exceeding three. If $\mathcal{T}$ is an edge-sum tree with the properties (i) and (ii) below, then call $\mathcal{T}$ a $(d, c; n)$-edge-sum tree. Furthermore, if $\mathcal{T}$ is a block tree or a 3-block tree, then call $\mathcal{T}$ a $(d, c; n)$-block tree or a $(d, c; n)$-3-block tree, respectively.

(i) $\mathcal{T}$ can be viewed as a rooted block tree of depth at most $d$, for some nonnegative integer $d$ that does not exceed $M_n$.
(ii) Each block of each node graph of $\mathcal{T}$ either has no cycle of length exceeding $N_n$ or is a cycle. Moreover, if $D(\mathcal{T}) = d$, and $B$ is a block of the root graph of $\mathcal{T}$ that is not a cycle, then $B$ has no cycle of length exceeding $c$, for some integer $c$ such that $1 < c \leq N_n$.

If $\mathcal{T}$ is a $(d, c; n)$-edge-sum tree, and each node graph different from the root graph is 2-connected, then call $\mathcal{T}$ a $(d, c; n)$-near-block tree. If $\mathcal{T}$ is a $(d, c; n)$-block tree or
a \((d,c;n)\)-3-block tree, and \(T\) is \(n\)-close, then call \(T\) a \((d,c;n)\)-close block tree or a \((d,c;n)\)-close 3-block tree, respectively. In particular, each \((0,c;n)\)-block tree is a \((0,c;n)\)-close block tree. Note that if \(2 \leq c' \leq c \leq N_n\), then each \((d,c';n)\)-edgesum tree is a \((d,c;n)\)-edgesum tree. Also, note that if \(0 \leq d' < d \leq M_n\), then each \((d',N_n;n)\)-edgesum tree is a \((d,c;n)\)-edgesum tree.

Now, we are ready to state the main result Theorem 7.1 of this section. The statement of Theorem 7.1 will be followed by several lemmas that will be used in its proof.

7.1. Theorem. Let \(T = (S, T)\) be a \((d,c;n)\)-close 3-block tree for some integer \(n\) exceeding three. Then one of the following holds.

\( (i) \) \( t(G(T)) < F(n) \), where \( F(n) = \frac{n^3(N_n+1)^3}{6} + \frac{2n^3(n(N_n+1))^3}{3} + \frac{N_n(N_n+1)}{2} \).

\( (ii) \) \( C_{n,n-2} \leq G(T) \) or \( C_{n,n-2} \leq G(T) \).

The first lemma, which is stated without proof, describes a well-known property of edge-cuts in connected graphs. In the lemma that follows, we shall show that, for each block \(B\) of \(G(T)\) that contains more than one edge, there is a \((d,c;n)\)-block tree \(T_B\), called a block-tree reduction of \(B\) in \(T\), such that \(G(T_B) = B\).

7.2. Lemma. If \(G\) is a connected graph and \(S\) is an \(xy\)-edge-cut in \(G\), then \(G\setminus S\) is made up of two components, \(C_x\) containing \(x\) and \(C_y\) containing \(y\).

7.3. Lemma. Let \(T = (S, T)\) be a \((d,c;n)\)-edgesum tree whose composition is \(G\), where \(n\) is an integer that exceeds 3. If \(B\) is a block of \(G\) that contains at least one edge, then there is a \((d,c;n)\)-edgesum tree, namely, \(T_B\), whose composition is \(B\). In particular, if \(B\) is 2-connected, then there is a \((d,c;n)\)-edgesum tree \(T_B\) whose composition is \(B\). Moreover, if \(e \in E(B)\) belongs to the root graph of \(T_G\), then \(e\) belongs to the root graph of \(T_B\).

Proof. If \(B\) is a link-edge of \(G\) or a cycle of \(G\), then it follows that \(T_B = (\{B\}, K_1)\) is a \((0,2,n)\)-edgesum tree. It is trivial that \(T_B\) satisfies the remaining conditions stated in the lemma. So, for the remainder of the proof, we may assume that \(B\) is 2-connected and not a cycle.

If \(D(T) = 0\), then \(T_G = (\{G\}, K_1; n)\), and \(T_G\) is a \((0,c;n)\)-edgesum tree. It follows that \(B\) has no cycles of length exceeding \(c\). Then \(T_B = (\{B\}, K_1; n)\) is a \((0,c;n)\)-block tree. The remaining condition of the lemma is satisfied since \(D(T_B) = 0\).

Now, we may assume that \(d\) is a positive integer, and that the lemma holds for all \((d-1,N_n; n)\)-edgesum trees. Consider \(T = T_G/(E(G) - E(B))\). By Lemma 4.6, \(G(T) = G/(E(G) - E(B))\), which is 1-isomorphic to \(B\). Let \(ξ\) denote the root of \(T\), and let \(H\) denote the root graph of \(T\). Let us consider the star \(S_ξ\) of \(ξ\) at \(H\) as defined earlier in this section and use the notation introduced in that definition.

If there is some \(j \in [m]_\ast\) so that the set of node graphs of \(T_ξ/(\{ε_i : i \in [m]_\ast - \{j\}\})\) is made up of \(K^3\) and a graph \(H_0\) whose set of edges consists of a single edge, namely \(h_j\), then consider the edgesum tree \(T_j^\ast = T/(E(T) - E(T_j))\). The tree \(T_j^\ast\), which is isomorphic to \(T_j\), is obtained by contracting \(E(T) - E(T_j)\) in \(T\) to the node \(ξ_j\); let us view \(T_j\) as being rooted at \(ξ_j\). Since \(T_j^\ast\) is a partial composition of \(T\), it follows that \(G(T_j^\ast) = G(T) ≅ B\). The set of node graphs of \(T_j^\ast\) is obtained from the set of node graphs of \(T_j\) by replacing the root graph \(H_j\) of \(T_j\) by \(H_j/k_j\); if \(h_j\) is a loop, and by \(H_j\setminus k_j\) if \(h_j\) is not a loop. It follows that \(T_j^\ast\) is a rooted edgesum tree of depth less than \(d\) whose composition is 1-isomorphic to \(B\), and that each
block of each node graph of $\mathcal{T}'_j$ either lacks a cycle of length exceeding $N_n$ or is a cycle. Hence, $\mathcal{T}'_j$ is a $(d - 1, N_n; n)$-edge-sum tree, and, by hypothesis, there is a $(d, c; n)$-block tree $\mathcal{T}_B$ of $B$, and the result holds.

Finally, we may assume that the node graph $H'_j$ of $\mathcal{T}_J/\{\varepsilon : i \in [m]_+ \setminus \{j\}\}$ that is not $K'$ has at least two edges (hence, at least one unlabeled edge), for each $j \in [m]_+$. It follows that all edges of $K'$ belong to a single block of $K'$, for each $i \in [m]_+$; otherwise, $G(\mathcal{T})$ would have more than one block containing edges. For each $i \in [m]_+$, if $K'$ consists of a single edge, then contract the link $\varepsilon_i$ in $\mathcal{T}_i$. Let $\mathcal{T}' = (\mathcal{G}', \mathcal{T}')$ denote the resulting rooted edge-sum tree, and let $H'$ denote the root graph of $\mathcal{T}'$. It follows that each node graph $K'$ in $\mathcal{G}' - H'$ is a 2-connected graph with, perhaps, some isolated vertices. It then follows that $H'$ is a 2-connected graph with, perhaps, some isolated vertices; otherwise, $G(\mathcal{T})$ would have more than one block containing edges. Note that, for each node graph $K'$ in $\mathcal{G}' - H'$, the graph $K'$ is the composition of the edge-sum tree $\mathcal{T}_i$ rooted at $\xi_i$, for some $i \in [m]_+$, and the edge $k_i \in E(K')$ belongs to the root graph of $\mathcal{T}_i$. By hypothesis, for each node graph $K'$ in $\mathcal{G}' - H'$, there is a $(d - 1, N_n; n)$-block tree $\mathcal{T}_k'$ whose composition is $K'$ and whose root graph contains $k_i$. Consider the rooted edge-sum tree $\mathcal{T}'$ defined as follows. Let $H^* = H'[E(H')]$ be the root graph of $\mathcal{T}^*$, let the directed labeling of $H^*$ in $\mathcal{T}^*$ agree with the directed labeling of $H'$ in $\mathcal{T}'$, and let $\xi^*$ denote the root of the tree $T^*$ of $\mathcal{T}^*$. We obtain $T^*$ by connecting $\xi^*$ to the root of the tree of $\mathcal{T}_k'$, with a link, for each $K'$ in $\mathcal{G}' - H'$. If $h_i$ is a labeled edge of $H^*$, then there is a $(d - 1, N_n; n)$-block tree $\mathcal{T}_k'$, for some $K'$ in $\mathcal{G}' - H'$, whose root graph contains $k_i$. Assign the label $\varepsilon_i$ to $k_i$, and direct $k_i$ in $\mathcal{T}'$ so that its direction agrees with its direction in $\mathcal{T}'$. Note that $H^*$ is obtained by deleting all isolated vertices from $H'$. Also, note that $H'$ is obtained by edge-summation $H$ with graphs $K'$ each consisting of a single edge, which amounts to deleting or contracting an edge of $H$, depending on whether such a graph $K'$ is a link-edge or a loop. Lastly, note that $H$ is obtained by contracting a set of edges in the root graph $H_G$ of $\mathcal{T}_G$, and thus $H^* \leq M, H_G$. Since each block of $H_G$ is a cycle or contains no cycle of length exceeding $c$, and since $H'$ is a 2-connected graph with, perhaps, some isolated vertices, it follows that $H^*$ is a cycle, or $H^*$ is a block that contains no cycle of length exceeding $c$. It follows that $\mathcal{T}^*$ is a $(d, c; n)$-block tree whose composition is $B$, as required.

We shall prove Theorem 7.1 by induction on the indices $d$ and $c$. The next few lemmas will handle the details of certain steps of the induction in order to make the proof of Theorem 7.1 shorter and more readable.

7.4. Lemma. Let $\mathcal{T} = (\mathcal{G}, L_G, T)$ be a $(0, c; n)$-close block tree, for some integer $n$ exceeding 3. Then $t(G(\mathcal{T})) \leq \frac{d(c+1)}{2}$.

Proof. Since $D(T) = 0$, it follows that $\mathcal{G}$ contains only one node graph $H$, which is an unlabeled 2-connected graph, and $G(\mathcal{T}) = H$. Recall that $2 \leq c \leq N_n$. If $H$ is a cycle with at least 2 edges, then $t(H) = 2 < \frac{d(c+1)}{2}$. So, we may assume that $H$ is a 2-connected graph, each cycle of which has length at most $c$. By Corollary 3.6, $t(H) \leq \frac{d(c+1)}{2}$. ~\(\square\)

7.5. Lemma. Let $\mathcal{T} = (\mathcal{G}, L_G, T)$ be a $(d, c; n)$-close block tree whose root graph is a cycle of length at least $n$, for some integers $n$ and $d$ exceeding 3 and 0, respectively. Then one of the following holds.
(i) There is a set $S$ of at most $n-3$ edges in $G(T)$, so that if $B$ is a 2-connected block of $G(T)\setminus S$, for which $t(B) = t(G(T)\setminus S)$, then there is a $(d-1, N_n; n)$-block tree $T_B$ whose composition is $B$.

(ii) $t(G(T)) \leq n - 2$.

(iii) $C_{n, n-2} \leq m G(T)$.

Proof. Let $H$ denote the root graph of $T$, and let $\xi$ denote the root of $T$. The cycle $H$ has length $N$, for some integer $N \geq n$. We may assume that $V(H) = \{v_i : i \in [N]_+\}$ and that $E(H) = \{v_1v_2, v_2v_3, \ldots, v_{N-1}v_N, v_Nv_1\}$. Also, it will be convenient to think of $v_1$ as sometimes having the name $v_{N+1}$. Let us consider the star $T_*$ of $T$ at $H$ as defined earlier in this section and use the notation introduced in that definition.

Since $T$ is a block tree, it follows that $T_i$ and $T_*$ are block trees and $K^i$ is 2-connected, for each $i \in [m]_+$. It also follows that the labeled edge $k_i$ in $K^i$ is a link-edge, and we shall let $x_i$ and $y_i$ denote the endvertices of $k_i$, for each $i \in [m]_+$. It follows that there are $N$ distinct vertices in $G(T_*) = G(T)$ corresponding to the $N$ vertices of $V(H)$. For each $i \in [N]_+$, let the vertex in $G(T_*)$ corresponding to $v_i$ also be called $v_i$. Since the edges $h_i$ and $k_i$ are identified (and then deleted) when $x_i$ is contracted in $T_*$, the composition $G(T_*)$ is obtained from $H$ by replacing $h_i$ with $K^i \setminus k_i$ so that $x_i$ is identified with one endvertex of $h_i$ and $y_i$ is identified with the other endvertex of $h_i$ (as determined by the directed labeling of $T_*$), for each $i \in [m]_+$. Let $K^i_k$ denote the subgraph of $G(T_*) = G(T)$ that is isomorphic to $K^i \setminus k_i$ and replaces $h_i$ in $H$, for each $i \in [m]_+$. Note that, for each $i \in [m]_+$, the graphs $K^i_k$ and $K^i \setminus k_i$ are identical except that $x_i$ and $y_i$ in $K^i \setminus k_i$ are renamed in $K^i_k$ with the endvertices of $h_i$ in $H$. Figure 9 illustrates a typical $T_*$ and its composition. In this figure, the cycle whose vertex set is $\{v_i : i \in [6]_+\}$ is the root graph $H$ of $T_*$, and, for each $i \in [3]_+$, the node graph of $T_*$ containing $k_i$ is $K^i$.

![Figure 9: A typical T_ and its composition G(T_).](image)

First, let us assume that some edge $e$ of $H$ is not labeled by the directed labeling $L_G$ of $T$, and thus $e \in E(G(T))$. By shifting the indices of the vertices of $H$, we may assume that $e = v_1v_N$. If $e$ is deleted from $G(T)$, then, for each integer $i$ such that $1 < i < N$, the vertex $v_i$ is a cut-vertex of $G(T) \setminus e$. It follows that each unlabeled edge in $E(H) - e$, viewed as a subgraph of $G(T) \setminus e$, is a block of $G(T) \setminus e$,
and $K^i_k$ is a union of blocks of $G(\mathcal{T}) \setminus e$, for each $i \in [m]_+$. Since $K^i_k$ has at least one edge, for each $i \in [m]_+$, it follows that $t(G(\mathcal{T}) \setminus e) = \max\{t(K^i_k) : i \in [m]_+\}$. Let $l \in [m]_+$ be an index for which $t(G(\mathcal{T}) \setminus e) = t(K^i_k)$, and let $B$ be a block of $K^i_k$ for which $t(K^i_k) = t(B)$. If $|E(B)| = 1$, then it follows that each block of $G(\mathcal{T}) \setminus e$ is a single edge, and hence $t(G(\mathcal{T})) \leq |\{e\}| + t(B) = 2 \leq n - 2$. So, we may assume that $B$ has more than one edge, and hence $B$ is 2-connected. Note that $K^i_k \cong K^i_k \setminus k_l = G(\mathcal{T}_l \setminus k_l)$. Since $\mathcal{T}_l \setminus k_l$ is a $(d - 1, N_n; n)$-edge-sum tree, by Lemma 7.3, there is a $(d - 1, N_n; n)$-block tree whose composition is $B$, as required.

For the rest of the proof, we may assume that each edge of $H$ is labeled by $L_G$, and consequently $m = N$. By an appropriate permutation of $[N]_+$ applied to the index $i$ in $e_i$, $h_i$, $k_i$, $\mathcal{T}_i$, $T_i$, $K^i$, and $K^i_k$, we may assume that $h_i = v_{i,v_{i+1}}$ in $H$.

For the next case, which is similar to the first, let us assume that there is an index $l \in [N]_+$ for which $K^l$ has an $x_l\mathcal{P}$-edge-cut $S^0_N$ containing at most $n - 2$ edges. By shifting the indices, we may assume that $l = N$. Clearly, $k_N \in S^0_N$. Let $S_N = S^0_N - k_N$. Then $S_N$ is made up of unlabeled edges, and $|S_N| \leq n - 3$. Note that $K^N_N \setminus S_N \cong K^N \setminus S^0_N$. Since $K^N$ is 2-connected, it follows from Lemma 7.2 that $K^N \setminus S^0_N$ is made up of two components, $C_x$ containing $x_N$ and $C_y$ containing $y_N$. Thus, $K^N_N \setminus S_N$ is made up of two components, $C_1$ containing $v_1$ and $C_N$ containing $v_N$, and $\{C_1, C_N\} \neq \{C_x, C_y\}$ identical except for the names of $v_1$ and $v_N$ in $\{C_x, C_y\}$. It follows that $G(\mathcal{T}) \setminus S_N = G(\mathcal{T}_N) \setminus S_N$ is as in Figure 10.

![Figure 10. A typical $G(\mathcal{T}) \setminus S_N$.](image-url)

Note that, for each integer $i$ such that $1 < i < N$, the vertex $v_i$ is a cut-vertex of $G(\mathcal{T}) \setminus S_N$. Furthermore, if $C_1$ is not isomorphic to $K_1$, then $C_1$ is a union of bridges of $v_1$ in $G(\mathcal{T}) \setminus S_N$. Similarly, if $C_N$ is not isomorphic to $K_1$, then $C_N$ is a union of bridges of $v_N$ in $G(\mathcal{T}) \setminus S_N$. It follows that $K^i_k$ is a union of blocks of $G(\mathcal{T}) \setminus S_N$ for each $i \in [N - 1]_+$, and $C$ is a union of blocks of $G(\mathcal{T}) \setminus S_N$, for each element $C$ of $\{C_1, C_N\}$ that is not isomorphic to $K_1$. If there is an index $q \in [N - 1]_+$ such that $t(K^q_q) = t(G(\mathcal{T}) \setminus S_N)$, then let $B$ be a block of $K^q_q$ such that $t(B) = t(K^q_q)$. If $B$ consists of a single edge, then each block of $G(\mathcal{T}) \setminus S_N$ is a single edge, and hence, $t(G(\mathcal{T})) \leq |S_N| + t(B) \leq n - 2$. Consequently, as before, we may assume that $B$ is 2-connected. Recall that $K^q_q \cong K^q \setminus k_q = G(\mathcal{T}_q \setminus k_q)$. Since $\mathcal{T}_q \setminus k_q$ is a $(d - 1, N_n; n)$-edge-sum tree, by Lemma 7.3, there is a $(d - 1, N_n; n)$-block tree whose composition is $B$, as required. If there is no index $q \in [N - 1]_+$ for which $t(K^q_q) = t(G(\mathcal{T}) \setminus S_N)$, then $t(C_1 \cup C_N) = t(G(\mathcal{T}) \setminus S_N)$. Let $B$ be a block of $C_1 \cup C_N$ for which $t(B) = t(C_1 \cup C_N)$. As before, $t(G(\mathcal{T})) \leq |S_N| + t(B) \leq n - 2$ if $B$ consists of an edge; so we may assume that $B$ is 2-connected. Note that
$C_1 \cup C_N \cong K^{N-1} \setminus S_N^0 \Rightarrow G(T_N \setminus S_N^0)$. Since $T_N \setminus S_N^0$ is a $(d - 1, N_n; n)$-edge-sum tree, by Lemma 7.3, there is a $(d - 1, N_n; n)$-block tree whose composition is $B$, as required.

For the final case, let us assume, for each $i \in [N]_+$, that every $x_i y_i$-edge cut in $K^i$ has at least $n - 1$ edges. The following holds for each $i \in [N]_+$. Let $S_i^0$ be an $x_i y_i$-edge cut in $K^i$. Clearly, $k_i \in S_i^0$, and the edges in $S_i^0 - k_i$ are unlabeled in $K^i$. Let $S_i = S_i^0 - k_i$. Then $|S_i| \geq n - 2$. By Lemma 7.2, $K_i \setminus S_i^0 \cong K_i^1 \setminus S_i$ consists of two components, $C_{i1}$ containing $v_i$ and $C_{i2}$ containing $v_i + 1$. It is straightforward that $S_i$ is a $v_i v_i + 1$-edge cut of $K_i^1$. From this we show that $C_i \cup C_i^2 \cup s$ is connected, for each $s \in S_i$, in particular, each $s \in S_i$ has one endvertex in $V(C_{i1})$ and the other endvertex in $V(C_{i2})$. Next, we show that a multi-edge of size at least $n - 2$ is a minor of $K_i^1$.

Consider $K_i^1 / E(C_{i1} \cup C_{i2})$, for any $i \in [N]_+$. We can see that this graph is isomorphic to a multi-edge of size at least $n - 2$ with endvertices $v_i$ and $v_i + 1$ as follows. When $E(C_{i1})$ is contracted in $K_i^1$, we may identify all of the vertices of $C_{i1}$ to $v_i$. Similarly, when $E(C_{i2})$ is contracted in $K_i^1$, we may identify all of the vertices of $C_{i2}$ to $v_i + 1$. Since $C_{i1}$ and $C_{i2}$ are disjoint, $v_i$ and $v_i + 1$ are distinct vertices in $K_i^1 / E(C_{i1} \cup C_{i2})$. Hence, in $K_i^1 / E(C_{i1} \cup C_{i2})$, one endvertex of $s$ is $v_i$ and the other endvertex of $s$ is $v_i + 1$, for each $s \in S_i$. Consequently, $K_i^1 / E(C_{i1} \cup C_{i2})$ is a multi-edge of size at least $n - 2$ with endvertices $v_i$ and $v_i + 1$, for each $i \in [N]_+$.

Since a multi-edge of size at least $n - 2$ with endvertices $v_i$ and $v_i + 1$ is a minor of $K_i^1$, for each $i \in [N]_+$, it follows that $C_{N_n} \leq G(T)$. Since $N \geq n$, it follows that $C_{n, n-2} \leq G(T)$, as required.

In Lemma 4.6 we saw that, given an edge-sum tree $T$ and disjoint sets $C$ and $D$ of edges in $G(T)$, contracting each edge of $C$ in its appropriate node graph and deleting each edge of $D$ in its appropriate node graph is equivalent to first taking the composition of $T$ and then performing contractions on the edges of $C$ and deletions on the edges of $D$. In order to prove the next lemma, we would like, in some sense, to be able to perform a contraction or a deletion on a labeled edge from the root node of a near-block tree $T$ and describe the effect of this on $G(T)$. This is described more precisely below.

Let $T$ be a near-block tree of depth at least 1, and let $h$ be a labeled edge in the root graph $H$ of $T$. Let $\varepsilon$ denote the link of the tree $T$ of $T$ with which $h$ is labeled, and let $k$ denote the other edge that is labeled $\varepsilon$. Let $T_k$ denote the restriction of $T$ induced by the component $T_k$ of $T \setminus \varepsilon$ that does not contain the root $\xi$ of $T$, and let $K = G(T_k)$. It follows that $k \in E(K)$, and, since $T_k$ is a block tree, $K$ is 2-connected. Hence, $k$ is a link-edge, and thus has distinct endvertices $x_k$ and $y_k$. Since $K$ is 2-connected, it has a cycle $C_k$ containing $k$ and an $x_k y_k$-edge cut $D_k$ containing $k$. Let $C_0 = E(C_k) - k$ and $D_0 = D_k - k$. Consider the partial composition $T_k / E(T_k)$ of $T$. The node graph of $T_k / E(T_k)$ that corresponds to the endnode of $\varepsilon$ that is not $\xi$ is $K$, and $K$ (viewed as a node graph of $T_k / E(T_k)$) has exactly one labeled edge $k$. By symmetry, we may assume that the $x_k$ and $y_k$ are the tail and head, respectively, of $k$.

First, let us consider $T_{1/\varepsilon} = (T_k / E(T_k)) / C_0$. Since the edges of $C_0$ form an $x_k y_k$-path in $K$, the labeled edge $k$ is a loop in the node graph $K / C_0$ of $T_1 / \varepsilon$. Now consider $T_{1/\varepsilon}$. It is straightforward that $T_1$ and $T_{1/\varepsilon}$ are the same, except that the link $\varepsilon$ is contracted to $\xi$ in $T_{1/\varepsilon}$, and the node graph in $T_1 / \varepsilon$ corresponding to $\xi$ is $H_1 = H \oplus_2 (K / C_0)$. Since $k$ is a loop in $K / C_0$, it follows from the definition of
edge-summing that $H_1$ is 1-isomorphic to the disjoint union of $H/h$ and $K/C_k$. Also note that $K/C_k$, viewed as a subgraph of $H_1$, has no labeled edges and is a union of blocks of $H_1$ (provided that $K/C_k$ has at least one edge). It is straightforward that $G(T_j)$ is 1-isomorphic to the disjoint union of $K/C_k$ and $G(T_j)$, where $T_j$ is obtained by contracting $h$ in the root graph $H$ of the restriction $T = V(T_k)$ of $T$. Note that $K/C_k$ is the composition of $T_{\mathcal{T}}/C_k$. Let us abbreviate $T_{\mathcal{T}}/C_k$ as $T_{\mathcal{T}}/C_k$. It follows from the way that $T_1$, $T_j$, and $T_{\mathcal{T}}/C_k$ are defined that $G(T_{\mathcal{T}}/C_k)$ is 1-isomorphic to the disjoint union of the compositions of $T_{\mathcal{T}}$, and $T_{\mathcal{T}}/C_k$. Thus, $t(G(T_{\mathcal{T}}/C_k)) = t(G(T_{\mathcal{T}}/C_k)) = \max\{t(G(T_{\mathcal{T}})), t(G(T_{\mathcal{T}}/C_k))\}$, and hence $t(G(T_{\mathcal{T}})) \leq |C_k| + \max\{t(G(T_{\mathcal{T}})), t(G(T_{\mathcal{T}}/C_k))\}$. Note that $T_{\mathcal{T}}/C_k$ is an edge-sum tree of depth less than $D(T)$ and that $T_j$ is a near-block tree. So let us say that we can essentially contract a labeled edge $h$ in the root graph $H$ of a near-block tree $T$ by contracting $C_k$ in $T$, and, after essentially contracting $h$, it is sufficient to consider $\{T_j, T_j/C_k\}$, as described above. The process of essentially contracting the labeled edge $h$ in $H$ in $T$ is illustrated in Figure 11.

![Figure 11](image)

Figure 11. $T_j$ and $G(T_j/C_k) = K/C_k$ are obtained by essentially contracting $h$.

Now, consider $T_2 = (T/E(T_k)) \setminus D_0$. Since $D_k$ is an $x_k/y_k$-edge-cut in $K$, it follows from Lemma 7.2 that $k$ is a cut-edge of $K \setminus D_0$ whose deletion results in components $K_{x_k}$ and $K_{y_k}$ containing $x_k$ and $y_k$, respectively (hence, $K \setminus D_k = K_{x_k} \cup K_{y_k}$). It follows that $T_2$ and $T_2/E$ are the same, except that $E$ is contracted to $\xi$ in $T_2/E$, and the node graph in $T_2/E$ corresponding to $\xi$ is $H_2 = H \boxplus_2 (K \setminus D_k)$. Note that whether $h$ is a loop or a link-edge, $H_2 \cong_1 (H \setminus h) \cup (K \setminus D_k)$. Also note that each component $K'$ of $K \setminus D_k$, viewed as a subgraph of $H_2$, is an unlabeled union of blocks of $H_2$, provided that $E(K') \neq \emptyset$. It follows that $G(T_2/E) \cong_1 (K \setminus D_k) \cup G(T_{\mathcal{T}})$, where $T_{\mathcal{T}}$ is obtained by deleting $h$ from $T = V(T_k)$. Note that $K \setminus D_k = G(T_{\mathcal{T}} \setminus D_k)$, and abbreviate $T_{\mathcal{T}} \setminus D_k$ as $T_{\mathcal{T}}/D_k$. It follows from the way that $T_2$, $T_{\mathcal{T}}$, and $T_{\mathcal{T}}/D_k$ are defined that $G(T \setminus D_0) \cong_1 G(T_{\mathcal{T}}) \cup G(T_{\mathcal{T}}/D_k)$. Thus, $t(G(T \setminus D_0)) \leq |D_0| + t(G(T) \setminus D_0) = |D_0| + t(G(T \setminus D_0) = |D_0| + \max\{t(G(T_{\mathcal{T}})), t(G(T_{\mathcal{T}}/D_k))\}$. Note that $T_{\mathcal{T}}/D_k$ is an edge-sum tree of depth less than $D(T)$ and that $T_{\mathcal{T}}$ is a near-block tree. So we can essentially delete a labeled edge $h$ from the root graph $H$ of a near-block tree $T$ by deleting $D_0$ from $T$, and, after essentially deleting $h$, we may consider $\{T_{\mathcal{T}}, T_{\mathcal{T}}/D_k\}$,
as described above. The process of essentially deleting a labeled link-edge \( h \) from \( H \) in \( \mathcal{T} \) is illustrated in Figure 12.

\[
\begin{array}{ccc}
\includegraphics[width=0.3\textwidth]{figure12a} & \includegraphics[width=0.3\textwidth]{figure12b} & \includegraphics[width=0.3\textwidth]{figure12c}
\end{array}
\]

\textbf{Figure 12.} \( \mathcal{T}\backslash h \) and \( G(\mathcal{T}\backslash D_k) = K\backslash D_k \) are obtained by essentially deleting \( h \).

Finally, we extend the definition to disjoint sets \( C \) and \( D \) of labeled edges in the root graph \( H \) of a near-block tree \( \mathcal{T} \) so that we essentially contract \( C \) and essentially delete \( D \). Consider the star \( \mathcal{T}_s \) of \( \mathcal{T} \) at \( H \). Recall that \( \{ h_i : i \in [m]_+ \} \) is the set of labeled edges in the root graph \( H \) of \( \mathcal{T} \). Since \( \mathcal{T}_s \) is a block tree, it follows that \( \mathcal{T}_s \) is 2-connected, for each \( i \in [m]_+ \). Hence, \( K^i \) contains a cycle \( C_{k_i} \) containing \( k_i \) and an \( x_i, y_i \)-edge-cut \( D_{k_i} \), where \( x_i \) and \( y_i \) are the tail and head, respectively, of \( k_i \) as determined by the directed labeling of \( \mathcal{T}_s \) for each \( i \in [m]_+ \). Note that there are subsets \( I_C \) and \( I_D \) of \([m]_+\) so that \( C = \{ h_i : i \in I_C \} \) and \( D = \{ h_i : i \in I_D \} \). Let \( C_i = C_{k_i} - k_i \) for each \( i \in I_C \), and let \( D_i = D_{k_i} - k_i \) for each \( i \in I_D \). Let \( \mathcal{T}_{i|C_{k_i}} = \mathcal{T}_{i|C_{k_i}} \) for each \( i \in I_C \), let \( \mathcal{T}_{i|D_{k_i}} = \mathcal{T}_{i|D_{k_i}} \) for each \( i \in I_D \), and let \( \mathcal{T}_{i|C_{k_i}} \) be the near-block tree that is obtained from the restriction \( \mathcal{T} - \bigcup_{i \in I_C \cup I_D} V(T_i) \) of \( \mathcal{T} \) by replacing its root graph \( H \) with \( H \backslash C \). Let us consider the collection \( \mathcal{I} = \{ \mathcal{T}_{i|C_{k_i}} \} \cup \{ \mathcal{T}_{i|D_{k_i}} : i \in I_C \} \cup \{ \mathcal{T}_{i|D_{k_i}} : i \in I_D \} \) of edge-sum trees. It is straightforward that the disjoint union of the compositions of the elements of \( \mathcal{I} \) is 1-isomorphic to \( G(\mathcal{T}) \backslash \bigcup_{i \in I_C} C_i \cup \bigcup_{i \in I_D} D_i \). It follows that \( t(G(\mathcal{T})) \leq |\bigcup_{i \in I_C} C_i| + |\bigcup_{i \in I_D} D_i| + \max \{ t(G(\mathcal{I})): \mathcal{I} \in \mathcal{I} \} \). So let us say that we can essentially contract \( C \) in and essentially delete \( D \) from the root graph \( H \) of a near-block tree \( \mathcal{T} \) by contracting \( \bigcup_{i \in I_C} C_i \) in and by deleting \( \bigcup_{i \in I_D} D_i \) from \( \mathcal{T} \).

After essentially contracting \( C \) and essentially deleting \( D \), it is sufficient to consider \( \mathcal{I} \), as described above.

\textbf{7.6. Lemma.} Let \( \mathcal{T} = (\mathcal{S}, \mathcal{T}) \) be a \((d, c; n)\)-close block tree whose root graph \( H \) contains no cycle of length exceeding \( c \), for some integers \( n, c \), and \( d \) exceeding, respectively, \( 3, 1, \) and \( 0 \), and assume that each edge of \( H \) is labeled. Then one of the following holds.

(i) There are disjoint subsets \( E_1 \) and \( E_2 \) of \( E(G(\mathcal{T})) \) containing fewer than \( 2^{n^2} \) edges and \( 2^{n^2} - 1 \) edges, respectively, so that if \( B \) is a 2-connected block of \( G(\mathcal{T}) \backslash E \) for which \( t(B) \geq t(G(\mathcal{T}) \backslash E) \), then there is an \((d, c - 1; n)\)-block tree if \( c \geq 3 \) or an \((d - 1, N; n)\)-block tree if \( c = 2 \), whose composition is \( B \).
(ii) \( t(G(\mathcal{J})) < \frac{c^3}{8} + 2n - 1 + c^2 n^2 + 1 \).
(iii) \( C_{n.n_{-2}} \subseteq G(\mathcal{J}) \).

Proof. Let \( \mathcal{J}_s \) be the star of \( \mathcal{J} \) at \( H \). For each \( i \in [n]_+ \), we assign a weight of \( s \) or \( l \) to \( h_i \in \mathcal{E}(H) \) as follows. If every cycle in \( K_i \) contains \( k_i \) length exceeding \( n(n-1) \), then let the weight \( w(h_i) \) be \( l \); otherwise, let \( w(h_i) = s \).

Let \( C \) be a longest cycle of \( H \). Clearly, \( |E(C)| = c \). For each pair \( \{u, v\} \) of vertices of \( C \), let \( P_{uv} \) be a \( uv \)-path in \( H \) made up of edges weighted \( s \) such that \( V(P_{uv}) \cap V(C) = \{u, v\} \), if such a path exists; otherwise, let \( P_{uv} \) be the subgraph of \( H \) made up of the vertices \( u \) and \( v \). Let \( P_s = \bigcup_{\{u,v\} \in V(C)} P_{uv} \), and let \( F_s \) be a spanning forest of \( P_s \), and hence \( |E(F_s)| \leq |E(P_s)| \). Note that if \( P_{uw} \) is a path, then the length of \( P_{uw} \) is at most the distance between \( u \) and \( v \) in \( C \), since \( C \) is a longest cycle in \( H \). It follows that

\[
|E(F_s)| \leq \begin{cases} c \sum_{i=1}^{c_3} i = c \cdot \frac{c^3 - 1}{2} < \frac{c^3}{8}, & \text{if } n \text{ is odd}; \\
2 \sum_{i=1}^{c_3} i + \frac{c^3}{2} \cdot \frac{c}{2} = c \cdot \frac{c^3 - 1}{2} + c^2 \cdot \frac{c}{8} < \frac{c^3}{8}, & \text{if } n \text{ is even}. 
\end{cases}
\]

Let \( I_F \) be the set of indices in \([n]_+\) for which \( h_i \in F_s \). For each \( i \in I_F \), let \( C_{k_i} \) be a cycle in \( K_i \) containing \( h_i \) whose length is at most \( n(n-1) \), and let \( C_i = E(C_{k_i}) \). Then \( C_i \) consists of unlabeled edges for each \( i \in I_F \). Let us essentially contract \( F_s \) in \( H \) by contracting \( C_s = \bigcup_{i \in I_F} C_i \) in \( \mathcal{J} \). Note that \( |C_s| \leq \frac{c^3}{8} \cdot n(n-1) < \frac{c^3 n^2}{8} \).

So, after contracting fewer than \( \frac{c^3 n^2}{8} \) edges in \( \mathcal{J} \), we may consider the collection \( \mathcal{F} = \{ \mathcal{J}_{/k_i} \} \cup \{ \mathcal{J}_{/C_{k_i}} : i \in I_F \} \) of edge-sum trees. Given any non-empty collection \( \mathcal{U} \) of edge-sum trees, let \( G(\mathcal{U}) \) denote the disjoint union of the compositions of the elements of \( \mathcal{U} \). It follows that \( G(\mathcal{J}/C_2) = G(\mathcal{J})/C_s \cong G(\mathcal{J}) \). Note that \( \mathcal{J}_{/k_i} \) is a \((d, c; n)\)-near-block tree, and \( \mathcal{J}_{/C_{k_i}} \) is a \((d - 1, N_n ; n)\)-edge-sum tree, for each \( i \in I_F \).

Let \( V' \) denote the set of vertices in the root graph \( H' = H/F_s \) of \( \mathcal{J}' = \mathcal{J}_{/F_s} \) corresponding to \( V(C) \) in \( H \). Clearly \( |V'| \leq |V(C)| = c \). We consider the cases when \( |V'| = 1 \) and when \( 1 < |V'| \leq c \) separately.

First, assume that \( |V'| = 1 \). Then, the length of each cycle in \( H' \) is at most \( c - 1 \). We can see this as follows. If \( C' \) is a longest cycle of \( H' \) different from \( C \), then, by Corollary 3.5, \( C \) and \( C' \) have at least two vertices in common. When \( F_s \) is contracted in \( H \), the vertices \( v_1 \) and \( v_2 \) are identified to a single vertex, and thus the subgraph of \( H' \) corresponding to \( C' \) is an edge-disjoint union of cycles of length less than \( c \). It follows that \( \mathcal{J}' = (d, c; 1; n)\)-near-block tree if \( c > 2 \). If \( c = 2 \), then, since \( H \) is a 2-connected graph, each block of \( H' \) is a loop. It follows that \( G(\mathcal{J}') = 1\)-isomorphic to \( \bigcup_{i \in I_H} G(\mathcal{J}_{/k_i}) \), where \( I_H = \{ i : h_i \in H' \} \). If \( c > 2 \), then let \( \mathcal{I}_0 = \mathcal{I} \); if \( c = 2 \), then let \( \mathcal{I}_0 = (\mathcal{I} - \{ \mathcal{J}' \}) \cup \{ \mathcal{J}_{/k_i} : i \in I_H \} \). Note that \( \mathcal{V}_{/k_i} \) is a \((d - 1, N_n, n)\)-edge-sum tree, for each \( i \in I_H \). It follows that \( \mathcal{I}_0 \) is made up of a number of \((d - 1, N_n, n)\)-edge-sum trees and, if \( c > 2 \), it consists of a single \((d, c; 1; n)\)-near-block tree. It is straightforward that \( G(\mathcal{J}/C_s) \cong G(\mathcal{I}_0) \).

If \( t(G(\mathcal{J}/C_s)) = 1 \), then \( t(G(\mathcal{J}/C_s)) \leq |C_s| + t(G(\mathcal{J}/C_s)) < \frac{c^3}{8} + 1 \), in which case the result follows. If \( t(G(\mathcal{J}/C_s)) = t(G(\mathcal{J}/C_s)) > 1 \), then there is a 2-connected block \( B \) for which \( t(B) = t(G(\mathcal{J}/C_s)) \). Since \( G(\mathcal{J}/C_s) \cong G(\mathcal{I}_0) \), the block \( B \) is isomorphic to a block of \( G(\mathcal{I}_0) \), for some \( \mathcal{I}_0 \in \mathcal{I}_0 \). By Lemma 7.3, there is a \((d, c; 1; n)\)-block tree or, if \( c = 2 \), a \((d - 1, N_n, n)\)-block tree whose composition is \( B \). So, if we let \( E_1 = C_s \) and \( E_2 = \emptyset \), the lemma follows.

Now, let us consider the case in which \( 1 < |V'| \leq c \). Note that \( H' \) is connected since \( H \) is connected. So, for each pair \( \{u, v\} \) of vertices in \( V' \), there is a \( uv \)-path
in $H'$, but we can see that no $uw$-path in $H'$ consists only of edges weighted $s$, as follows. If there were a $uv$-path in $H'$ consisting only of edges weighted $s$, then $H$ would contain, for some $\{u_0, v_0\} \subseteq V(C)$, a $u_0v_0$-path consisting only of edges weighted $s$. It follows that $F_s$ would contain a $u_0v_0$-path; consequently, $u_0$ and $v_0$ would be identified to the same vertex when contracting $F_s$ in $H$; a contradiction. So, for each pair $\{u, v\}$ of vertices in $V'$, each $uv$-path in $H'$ contains an edge weighted $l$. Then $H'$ contains a $uv$-edge-cut consisting only of edges weighted $l$, for each pair $\{u, v\}$ of vertices in $V'$.

If $|S_{uv'}| \geq n$, for some pair $\{u', v'\}$ of vertices of $V'$, then consider the restriction $T_{uv'}$ of the star $(T')_st$ of $T'$ induced by $\{e_i : i \in I_{uv'}\}$, where $i \in I_{uv'}$ if and only if $h_i \in S_{uv'}$. By Lemma 4.3, $G(T') \leq m G((T')_s) = G(T')$. Since the weight of $h_i$ is $l_i$, each cycle in $K'$ using $h_i$ has length exceeding $n(n - 1)$, for each $i \in I_{uv'}$. Let $C_{h_i}$ be a cycle in $K'$ using $h_i$, for each $i \in I_{uv'}$. Consider $T_{uv} = T' \cup \bigcup_{i \in I_{uv'}} (E(K') - E(C_{h_i}))$. It follows that $G(T_{uv})$ is obtained from $H'$ by subdividing the edge $h_i$ with $|E(C_{h_i})| - 2$ new vertices (that is, at least $n(n - 1) - 1$ new vertices), for each $i \in I_{uv'}$, and adding, perhaps, some isolated vertices. Let $P_i$ denote the path obtained by subdividing $h_i$, for each $i \in I_{uv'}$, as just described. It follows that $H' \setminus S_{uv'}$ and $G(T_{uv}) \setminus \bigcup_{i \in I_{uv'}} P_i$ are identical, except, perhaps, for additional isolated vertices in $G(T_{uv}) \setminus \bigcup_{i \in I_{uv'}} P_i$. By Lemma 7.2, $H' \setminus S_{uv'}$ consists of two components $C_{uv'}$ containing $v'$ and $C_{uv'}$ containing $v'$. Since $S_{uv'}$ is a $u'v'$-edge-cut in $H'$, exactly one endvertex $u_i$ of $h_i$ lies in $C_{uv'}$ and the other endvertex $v_i$ of $h_i$ lies in $C_{uv'}$, for each $i \in I_{uv'}$, and hence, in $G(T_{uv}) \setminus \bigcup_{i \in I_{uv'}} P_i$, for each $i \in I_{uv'}$, one endvertex of $P_i$ lies in $C_{uv'}$, and the other endvertex of $P_i$ lies in $C_{uv'}$. Let us contract $C_{uv'}$ to a single vertex $u'$ and $C_{uv'}$ to a single vertex $v'$ in $G(T_{uv})$. Since $P_i$ is obtained by subdividing $h_i$ in $H'$, for each $i \in I_{uv'}$, it follows that $G_0 = G(T_{uv})/E(C_{uv'} \cup C_{uv'})$ is made up of $|S_{uv'}| \geq n$ pairwise internally vertex-disjoint $u'v'$-paths, each having length at least $n(n - 1)$, and, perhaps, some isolated vertices. Hence, $C_{uv'} \leq m G_0 \leq m G(T_{uv}) \leq m G(T') \leq m G(T) \leq m G(T)$, and the lemma holds.

It remains to consider the case when $|S_{uv'}| < n$, for each pair $\{u, v\} \subseteq V'$. Let $S_i = \bigcup_{\{u, v\} \subseteq V, S_{uv}}$, let $I_S = \{i \in I_i : h_i \in S\}$, and let $x_i$ and $y_i$ denote the endvertices of $h_i$ in $K'$. Since $T$ is n-close (hence, $T_{uv'}$ is n-close), and since the weight of $h_i$ is $l$, there is an $x_i y_i$-edge-cut $D_{h_i}$ of size at most $n(n - 1)$ in $K'$, for each $i \in I_i$. Let $D_i = D_{h_i} - k_i$ for each $i \in I_i$. Let $D_i = D_{h_i} - k_i$ for each $i \in I_i$. Now, let us essentially delete $S_i$ from $H'$ by deleting $D_i = D_{h_i} - k_i$ for each $i \in I_i$. Note that $|D_i| \leq \sum_{i \in I_i} |D_i| < |S_i| n2^n2^n < n^{2^n} - 12^{2^n} n2^n < n^{2^n}$. So, after deleting fewer than $2n^n - c_2 n^2$ edges from $T'$, we may consider the collection $T' = \{T'_{I_S} : i \in I_S\}$ of edge-sum trees in place of $T'$. Note that $G(T') \leq m G(T')$. Let $T'' = (T' - \{T'\}) \cup T' = \{T'_{I_F \cup S_i} : i \in I_F\} \cup \{T'_{I_C \cup S_i} : i \in I_C\}$. It follows that $G(T')/C_{uv'} D_i \leq m G(T')$. Note that each element of $T'' - T'_{F \cup S}$ is a $(d - 1, N; n)$-edge-sum tree. Also, note that $T'' = T'_{F \cup S}$ is a $(d, c; n)$-near-block tree. In fact, we show below that the root graph $H'' = H/F_S S_i$ of $T''$ has only cycles of length less than $c$.

Let $C'$ be a longest cycle in $H$ different from $C$, if there is such a cycle. Since $T$ is a block tree, $H$ is a block tree, and consequently, by Corollary 3.5, $|V(C) \cap V(C')| \geq 2$. Let $\{v_1, v_2\} \subseteq V(C) \cap V(C')$. If $v_1$ and $v_2$ are identified to the same vertex when contracting $F_s$ in $H$, then the subgraph of $H'$ corresponding to $C'$ is an edge-disjoint union of cycles of length less than $c$. If $v_1$ and $v_2$ are identified to distinct
vertices that we shall call \(v_1\) and \(v_2\), respectively, in \(H'\), then \(v_1\) and \(v_2\) belong to distinct components of \(H''\) since \(S_{v_1v_2}\) was essentially deleted from \(\mathcal{T}'\) in forming \(\mathcal{T}''\) (hence, \(S_{v_1v_2}\) was deleted from \(H'\) in forming \(H''\)). It follows that \(S_{v_1v_2} \cap E(C')\) is non-empty. So, the subgraph of \(H''\) corresponding to \(C'\) contains fewer than \(c\) edges, and consequently \(C'\) is not a cycle of length \(c\) in \(H''\). Hence, \(H''\) has only cycles of length less than \(c\).

We conclude that \(\mathcal{T}''\) is a \((d,c-1;n)\)-near-block tree if \(c > 2\). Hence, \(\mathcal{T}''\) consists of one \((d,c-1;n)\)-near-block tree and a number of \((d-1,N_n;n)\)-edge-sum trees if \(c > 2\). If \(c = 2\), then each block of \(H''\) contains at most one edge. Let \(I_1\) and \(I_2\) denote the subsets of \([m]_+\) so that \(i \in I_1\) if and only if \(h_i\) is a loop in \(H''\), and \(i \in I_2\) if and only if \(h_i\) is a link-edge in \(H''\). It follows that \(G(\mathcal{T}'')\) is 1-isomorphic to \(\bigcup_{i \in I_1} G(\mathcal{T}' / k_i) \cup \bigcup_{i \in I_2} G(\mathcal{T}' \setminus k_i)\). If \(c > 2\), then let \(\mathcal{T}'_0 = \mathcal{T}''\); if \(c = 2\), then let \(\mathcal{T}'_0 = (\mathcal{T}'' - \{\mathcal{T}'\}) \cup \{\mathcal{T}' / k_i : i \in I_1\} \cup \{\mathcal{T}' \setminus k_i : i \in I_2\}\). It follows that \(G(\mathcal{T}' / C_n D_1) \cong G(\mathcal{T}'_0)\). Note that \(\mathcal{T}'_0\) consists of a number of \((d-1,N_n;n)\)-edge-sum trees and, if \(c > 2\), one \((d,c-1;n)\)-near-block tree.

If \(t(G(\mathcal{T}' / C_n D_1)) \leq 1\), then we have \(t(G(\mathcal{T}')) \leq |C_n| + |D_1| + t(G(\mathcal{T}' / C_n D_1)) < c^2 n^2 + 2n^2 - 1c^2 n^2 + 1\), in which case the lemma holds. So, we may assume that \(t(G(\mathcal{T}' / C_n D_1)) = t(G(\mathcal{T}')) / C_n D_1 > 1\). Then there is a 2-connected block \(B\) for which \(t(B) = t(G(\mathcal{T}' / C_n D_1))\). Since \(G(\mathcal{T}' / C_n D_1) \cong G(\mathcal{T}'_0)\), the block \(B\) is isomorphic to some block of \(G(\mathcal{T}'_0)\), for some \(\mathcal{T}'_0 \in \mathcal{T}'_0\). By Lemma 7.3, there is a \((d,c-1;n)\)-block tree or, if \(c = 2\), a \((d-1,N_n;n)\)-block tree whose composition is \(B\). So, if we let \(E_f = C_n\) and \(E_{\setminus} = D_1\), the proof is complete.

The next lemma is an extension of Lemma 7.6 in which some edges of the root graph of a \((d,c;n)\)-close block tree may be unlabeled.

**7.7. Lemma.** Let \(\mathcal{T} = (\mathcal{G}, \mathcal{T})\) be a \((d,c;n)\)-close block tree whose root graph contains no cycle of length exceeding \(c\), for some integers \(n, c, d\) exceeding 3, 1, and 0, respectively. Then one of the following holds.

(i) There are disjoint subsets \(E_f\) and \(E_{\setminus}\) of \(E(\mathcal{T})\) containing, respectively, fewer than \(\frac{c^2 n^2}{8}\) edges and fewer than \(2n^2 - 1c^2 n^2\) edges, so that if \(B\) is a 2-connected block of \(G(\mathcal{T}) / E_{\setminus} E_f\) whose type is \(t(G(\mathcal{T}) / E_{\setminus} E_f)\), then there is a \((d,c-1;n)\)-block tree if \(c \geq 3\) or a \((d-1,N_n;n)\)-block tree if \(c = 2\), whose composition is \(B\).

(ii) \(t(G(\mathcal{T})) < \frac{c^2 n^2}{8} + 2n^2 - 1c^2 n^2 + 1\).

(iii) \(c_{n,n-2} \leq m G(\mathcal{T})\), or \(c_{n,n-2} \leq m G(\mathcal{T})\).

**Proof.** We may assume that the root graph \(H\) of \(\mathcal{T}\) contains at least one unlabeled edge; otherwise, the desired result is immediate, by Lemma 7.6. Let \(E_0\) denote the set of unlabeled edges in \(H\). For each \(e \in E_0\), we can assign a direction and a new label \(e\) to \(e\), add a pendant link \(e\) to \(e\), at the root \(\xi\) of the tree \(T\) of \(\mathcal{T}\), let the node graph corresponding to \(\eta_e\) be a 2-cycle \(C_e\), and assign a direction and the label \(e\) to one of the edges of \(C_e\). Let \(\mathcal{T}\) denote the resulting \((d,c;n)\)-block tree, and let \(f_e\) denote the unlabeled edge of \(C_e\), for each \(e \in E_0\). It is evident that \(G(\mathcal{T}) \cong G(\mathcal{T})\). If \(\mathcal{T}\) is not \(n\)-close, then, by Corollary 6.3, an element of \(\{c_{n,n-2}, c_{n,n-2}\}\) is a minor of \(G(\mathcal{T}) \cong G(\mathcal{T})\), and the result follows. So we may assume that \(\mathcal{T}\) is \(n\)-close.

If \(t(G(\mathcal{T})) < \frac{c^2 n^2}{8} + 2n^2 - 1c^2 n^2 + 1\), or if \(c_{n,n-2} \leq m G(\mathcal{T})\), then the lemma holds, since \(G(\mathcal{T}) \cong G(\mathcal{T})\). Otherwise, by Lemma 7.6, there are disjoint subsets \(\bar{E}_f\) and \(\bar{E}_{\setminus}\) of \(E(G(\mathcal{T}))\) containing, respectively, fewer than \(\frac{c^2 n^2}{8}\) edges and fewer
than $2^n - c^2 n^2$ edges so that, if $B$ is a 2-connected block of $G(\mathcal{T})/\mathcal{E}_j \setminus \mathcal{E}_j$ for which $t(B) = t(G(\mathcal{T})/\mathcal{E}_j \setminus \mathcal{E}_j)$, then there is a $(d, c - 1; n)$-block tree if $c \geq 3$ or a $(d - 1, N_n; n)$-block tree if $c = 2$, whose composition is $B$. Note that when Lemma 7.6 is applied to $\mathcal{T}$, each edge in $E_0$ is weighted $s$ in $\mathcal{T}$, and the edges in $E_0$ correspond to edges of $H$ weighted $l$ in $\mathcal{T}$. Let $E_j = (E_\mathcal{J} - \{f_e : e \in E_0\}) \cup \{e : f_e \in E_\mathcal{J}\}$ and $E_\mathcal{J} = \mathcal{E}_j$. It is straightforward that $|E_j| = |\mathcal{E}_j|$ and $|E_\mathcal{J}| = |\mathcal{E}_\mathcal{J}|$, that $E_j$ and $E_\mathcal{J}$ are disjoint subsets of $E(G(\mathcal{T}))$, and that $G((\mathcal{T})/E \setminus E_\mathcal{J}) \cong G(\mathcal{T})/\mathcal{E}_j \setminus \mathcal{E}_\mathcal{J}$. The result follows. \hfill $\Box$

Now, we are ready to prove Theorem 7.1, whose proof uses several of the above lemmas.

**Proof of Theorem 7.1.** Let $\mathcal{T} = (\mathcal{G}, \mathcal{T})$ be a $(d, c; n)$-close 3-block tree. Recall that $n$ is an integer exceeding 3, $0 \leq d \leq 2(n - 1)$, and $2 \leq c \leq N_n$. We shall show that

$$t(G) \leq f(d) = d \sum_{i=1}^{N_n} \left( \frac{i^3 n^2}{8} + 2^n - 1 - i^2 n^2 \right) + \frac{N_n (N_n + 1)}{2},$$

or $C_{n,n-2} \leq m G(\mathcal{T})$, or $C_{n,n-2} \leq m G(\mathcal{T})$. Note that

$$f(d) = d \sum_{i=1}^{N_n} \left( \frac{i^3 n^2}{8} + 2^n - 1 - i^2 n^2 \right) + \frac{N_n (N_n + 1)}{2}$$

$$\leq 2(n - 1) \left( \frac{n^2}{8} \sum_{i=1}^{N_n} i^2 + 2^n - 1 - n^2 \sum_{i=1}^{N_n} i^2 \right) + \frac{N_n (N_n + 1)}{2}$$

$$< 2n \left( \frac{n^2 N_n^2 (N_n + 1)^2}{8} + \frac{2^n - 1 - n^2 N_n (N_n + 1) (2N_n + 1)}{6} \right) + \frac{N_n (N_n + 1)}{2}$$

$$< \frac{n^3 (N_n + 1)^4}{16} + \frac{2^n n^3 (N_n + 1)^3}{3} + \frac{N_n (N_n + 1)}{2} = F(n).$$

We proceed by induction on $d$ which includes within it induction on $c$. If $d = 0$, then, by Lemma 7.4, $t(G(\mathcal{T})) \leq \frac{c^3 + 1}{2} \leq \frac{N_n (N_n + 1)}{2} = \frac{N_n (N_n + 1)}{2} \cdot \sum_{i=1}^{N_n} \left( \frac{i^3 n^2}{8} + 2^n - 1 - i^2 n^2 \right) + \frac{N_n (N_n + 1)}{2}$, as required. For the remainder of the proof, let us assume that $d > 0$, and the result holds for each $d' \in [d - 1]$.

If the root graph of $\mathcal{T}$ is a cycle of length at least $n$, then, by Lemma 7.5, either $C_{n,n-2} \leq m G(\mathcal{T})$, or $t(G(\mathcal{T})) \leq n - 2 < 2^n < N_n < f(d)$, or there are a set $\mathcal{S}_\mathcal{J}$ of at most $n - 3$ edges in $G(\mathcal{T})$ and a $(d - 1, N_n; n)$-block tree $\mathcal{T}_B$ whose composition is a 2-connected block $B$ of $G(\mathcal{T}) \setminus \mathcal{S}_\mathcal{J}$, for which $t(B) = t(G(\mathcal{T}) \setminus \mathcal{S}_\mathcal{J})$. The first two of these alternatives imply the conclusion of the theorem, and so we may assume that the last condition listed holds. It follows that $B \leq m G(\mathcal{T})$ and that $t(G(\mathcal{T})) \leq |\mathcal{S}_\mathcal{J}| + t(B)$. If $\mathcal{T}_B$ is not $n$-close, then, by Corollary 6.3, $C_{n,n-2} \leq m G(\mathcal{T})$ or $C_{n,n-2} \leq m G(\mathcal{T})$, and the conclusion follows. So, we may assume that $\mathcal{T}_B$ is $n$-close. By the induction hypothesis, $t(B) \leq f(d - 1)$. It follows that $t(G(\mathcal{T})) \leq n - 3 + f(d - 1) < \sum_{i=1}^{N_n} \left( \frac{i^3 n^2}{8} + 2^n - 1 - i^2 n^2 \right) + f(d - 1) = f(d)$, as required.

We may assume for the remainder of the proof that the root graph of $\mathcal{T}$ contains no cycles of length exceeding $c$. By Lemma 7.7, either a graph in $\{C_{n,n-2}, C_{n,n-2}\}$ is a minor of $G(\mathcal{T})$, or $t(G(\mathcal{T})) < \frac{c^3 n^2}{8} + 2^n - 1 - c^2 n^2 + 1 < \frac{N_n^3 n^2}{8} + 2^n - 1 - N_n^2 n^2 +
\[ N_n \left( \frac{N_n + 1}{2} \right) < f(d), \] or conclusion (i) in Lemma 7.7 holds. The first two of these 
alternatives imply the conclusion of the theorem, and so we may assume that the 
last condition listed holds.

Now, we shall show that \( t(G(T)) \leq g(c) = \sum_{i=1}^{c} \left( \frac{3n^2}{8} + 2n^3 - 1^2 n^2 \right) + f(d - 1), \) or 
\( C_{n,n-2} \leq m \quad G(T), \) or \( C_{n,n-2} \leq m \quad G(T). \) Note that \( g(c) = \sum_{i=1}^{c} \left( \frac{3n^2}{8} + 2n^3 - 1^2 n^2 \right) + 
(d - 1) \sum_{i=1}^{N_n} \left( \frac{3n^2}{8} + 2n^3 - 1^2 n^2 \right) + N_n \left( \frac{N_n + 1}{2} \right) \leq f(d). \) Hence, it will follow that 
t(G(T)) \leq f(d), or \( C_{n,n-2} \leq m \quad G(T), \) or \( C_{n,n-2} \leq m \quad G(T). \) If \( c = 2, \) then there are 
disjoint sets \( E_f \) and \( E_\setminus \) containing fewer than \( n^2 \) and \( 2n^2 - 1^2 n^2 \) edges, respectively, 
in \( G(T) \) and a \( (d-1, N_n; n) \)-block tree \( T_B \) whose composition is a 2-connected block 
\( B \) of \( G(T)/E \setminus E_\setminus \) such that \( t(B) = t(G(T)/E \setminus E_\setminus). \) If \( T_B \) is not \( n \)-close, then, by 
Corollary 6.3, \( C_{n,n-2} \leq m \quad G(T) \) or \( C_{n,n-2} \leq m \quad G(T), \) and the conclusion follows. So, 
we may assume that \( T_B \) is \( n \)-close. By the induction hypothesis, \( t(B) \leq f(d - 1). \) It 
follows that \( t(G(T)) \leq |E_f| + |E_\setminus| + t(B) < n^2 + 2n^3 - 1^2 n^2 + f(d - 1) < \sum_{i=1}^{c} \left( \frac{3n^2}{8} + 
2n^3 - 1^2 n^2 \right) + f(d - 1) = g(2) \leq f(d), \) as required. So, let us assume that \( 3 \leq c \leq N_n \) and 
that \( t(G(\mathbb{U})) \leq g(c'), \) or \( C_{n,n-2} \leq m \quad G(\mathbb{U}), \) or \( C_{n,n-2} \leq m \quad G(\mathbb{U}) \) when \( \mathbb{U} \) is a 
\( (d,c'; n) \)-close block tree and \( c' \) satisfies \( 2 \leq c' < c. \)

There are disjoint subsets \( E_f \) and \( E_\setminus \) of \( G(T) \) containing fewer than \( \frac{3n^2}{8} \) edges 
and \( 2n^3 - 1^2 n^2 \) edges, respectively, and a \( (d,c - 1; n) \)-block tree \( T_B \) whose composition 
is a 2-connected block \( B \) of \( G(T)/E \setminus E_\setminus \) such that \( t(B) = t(G(T)/E \setminus E_\setminus), \) by 
Lemma 7.7. Again, by Corollary 6.3, if \( T_B \) is not \( n \)-close as before, \( C_{n,n-2} \leq m \quad G(T) \) 
or \( C_{n,n-2} \leq m \quad G(T), \) and the claim holds. So, we may assume that \( T_B \) is \( n \)-close. 
By the second induction hypothesis, \( t(B) \leq g(c - 1). \) It follows that 
t(G(T)) \leq |E_f| + |E_\setminus| + t(B) < \frac{3n^2}{8} + 2n^3 - 1^2 n^2 + g(c - 1) = g(c) \leq f(d), \) as 
required. The theorem follows.

Since each 2-connected graph with more than two edges can be decomposed into 
a unique 3-block tree, and since a 2-connected graph with at most 2 edges has very 
small type, the theorem below follows immediately on combining Corollary 4.4, 
Theorem 5.1, and Theorem 7.1.

7.8. Theorem. If \( G \) is a 2-connected graph such that
\[ t(G) \geq \frac{n^2(N_n + 1)^2}{16} + \frac{2n^3(N_n + 1)^3}{3} + \frac{N_n(N_n + 1)}{2} \]
for some integer \( n \) exceeding 3, then an element of \( \{ F_n, C_{n,n-2}, C_{n,n-2}^n \} \) is a minor 
of \( G. \) \( \square \)

Since the type of a general graph is the maximum of types of its blocks, Theorem 
7.8 immediately implies Theorem 1.7.

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