$M_1 = M_2$. For each $i$, let $N_i'$ be the $q$-cone of $N_i$ with apex $p$ and base $E(N_i)$.
For each $d$ in $\{a, b, c\}$, let the lines through $p$ and $d_1$ and through $p$ and $d_1'$ be
$\{d_1, d_2, \ldots, d_q, p\}$ and $\{d_1', d_2', \ldots, d_q', p\}$, respectively.

**Lemma 2.2.** In $M_i'$, suppose that both $\{a_i, a_i', b_k, b_i'\}$ and $\{b_k, b_i', c_m, c_i'\}$ are
circuits. Then so is $\{a_i, a_i', c_m, c_i'\}$.

**Proof.** The plane $P_{ab}$ of $N_i'$ spanned by $\{a_i, a_i', b_k, b_i'\}$ meets the line spanned by
$\{p, t\}$ in a single point, $t'$. Since $t'$ also lies in the plane $P_{pb}$ of $N_i'$ spanned by
$\{p, b_k, b'_i\}$, we deduce that $\{t', b_k, b'_i\} \subseteq P_{ab} \cap P_{pb}$, so $t', b_k,$ and $b'_i$ are collinear.
Similarly, $t', a_i,$ and $a'_i$ are collinear, and $t', c_m,$ and $c_i'$ are collinear. We deduce
that $\{a_i, a'_i, c_m, c'_i\}$ is a circuit of $M_i'$.

**Theorem 2.3.** $M_i'$ and $M_2'$ are non-isomorphic $q$-cones of $M_1$.

**Proof.** It suffices to show that $M_2'$ does not satisfy the condition in the preceding
lemma. In $N_2'$, consider the lines through $p$ and $t_{ab}$, through $p$ and $t_{ac},$ and through
$p$ and $t_{bc}$. Let $t_{ab}', t_{ac}',$ and $t_{bc}'$ be points of these lines different from $p$. The plane $P'$
spanned by $\{t_{ab}', t_{ac}', t_{bc}'\}$ meets the plane spanned by $\{p, a_i, a'_i\}$ in the line spanned
by $t_{ab}', t_{ac}',$ and this line contains a unique $a_i$ and a unique $a'_i$. Likewise, $P'$
contains a unique $b_k,$ a unique $b'_i,$ a unique $c_m,$ and a unique $c'_i$.

Let $t_{ab}'$ be a point on the line spanned by $p$ and $t_{ab}$ that is different from both
$p$ and $t_{ab}'. Then the plane $P''$ spanned by $\{t_{ab}', t_{ac}', t_{bc}'\}$ contains $\{c_m, c'_i\}$ and
$\{a_u, a'_v\}$ for some $u$ and $v$ distinct from $i$ and $j,$ respectively, where $t_{ab}' = t_{ac}' = a_u,$ and
$a_i'$ are collinear. Since $t_{ab}', t_{ac}', b_k,$ and $b_i'$ are collinear, the set $\{a_u, a'_v, b_k, b_i'\}$ spans
$\{t_{ab}', t_{ac}', t_{bc}', t_{bc}'\}$. But the last set has rank 4, so $\{a_u, a'_v, b_k, b_i'\}$ is not a circuit
of $M_1'$. However, both $\{a_u, a'_v, c_m, c'_i\}$ and $\{b_k, b'_i, c_m, c'_i\}$ are circuits of $M_1'$. We
conclude that $M_2'$ fails to satisfy the condition of the last lemma, so $M_2' \neq M_1'$.

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be the points of $N \setminus p$ on $L_1$. Then it is not difficult to check that, for each $i$ in \{1, 2, 3\}, the line through $(0, 1, b_i)^T$ and $(1, \alpha, a)^T$ meets $L_2$ and $L_3$ in $(1, 0, a - \alpha b_i)^T$ and $(1, 1, a + b_i - \alpha b_i)^T$, respectively (see Figure 1). Then, since $N \cong PG(2, 3)$, without loss of generality, $(0, 1, b_1)^T, (1, 0, a - \alpha b_2)^T,$ and $(1, 1, a + b_3 - \alpha b_3)^T$ are collinear. This implies that $(0, 1, b_2)^T, (1, 0, a - \alpha b_1)^T,$ and $(1, 1, a + b_3 - \alpha b_3)^T$ are collinear. The first of these two lines implies that $b_3(1 - \alpha) = b_2 - \alpha b_1$, while, by symmetry, the second implies that $b_2(1 - \alpha) = b_1 - \alpha b_2$. Combining these two equations gives $b_2 - \alpha b_1 = b_1 - \alpha b_2$, so $(1 + \alpha)b_2 = (1 + \alpha)b_1$. As $\alpha \in GF(q) \backslash GF(3)$, we deduce that $b_2 = b_1$; a contradiction.

Let $N_1$ and $N_2$ be the rank-3 matroids for which geometric representations are shown in Figure 2. For all prime powers $q \geq 4$, both $N_1$ and $N_2$ are $GF(q)$-representable. For each $i$ in \{1, 2\}, let $M_i = N_i \{a_i, a'_i, b_i, b'_i, c_i, c'_i\}$. Evidently
by its image under some automorphism of $GF(q)$. Moreover, we define $A_1$ and $A_2$ to be \textit{geometrically equivalent} $GF(q)$-representations of $M$ if the map that takes each column of $A_1$ to the corresponding column of $A_2$ is induced by an automorphism of the matroid corresponding to $PG(r - 1, q)$. Such an automorphism of $PG(r - 1, q)$ is a permutation of the set of subspaces that preserves dimension and inclusion. Equivalently, it is a permutation of the set of points of $PG(r - 1, q)$ that maps lines to lines. The last definition accounts for the name \textit{collineation} for such maps. It is a consequence of the Fundamental Theorem of Projective Geometry (see, for example, [1, p. 44] or [2, p. 655]) that, when $r \neq 2$, the representations $A_1$ and $A_2$ are algebraically equivalent if and only if they are geometrically equivalent. Thus, when $r \neq 2$, these two notions of equivalence coincide and it is conventional to refer to this common notion as simply \textit{equivalence} (or sometimes \textit{projective equivalence} [4]). However, when $r = 2$, the situation is less clear. The collineation group of $PG(1, q)$ is the symmetric group and therefore two representations may be geometrically equivalent without being algebraically equivalent. Although Oxley [5, pp.185-189] used “equivalent” to mean “geometrically equivalent”, we shall use \textit{equivalent} here to mean “algebraically equivalent”. To complete the picture, we note that there is yet another notion of equivalence: $A_1$ and $A_2$ are \textit{strongly equivalent} if $A_2$ can be obtained from $A_1$ by a sequence of the matrix operations described above without applying a field automorphism. Thus $A_1$ and $A_2$ are strongly equivalent if and only if there is a linear transformation $\sigma$ of $V(r, q)$ and a sequence $c_1, c_2, \ldots, c_n$ of non-zero elements of $GF(q)$ such that $\underline{w}^{(2)}_j = c_j \sigma(\underline{w}^{(1)}_j)$ for all $j$ where $\underline{w}^{(i)}_j$ is the $j$ th column of $A_i$. The same assertion holds for equivalence if we replace $\sigma$ by a semilinear transformation [5, p. 186].

Kung [4, p.102] notes that the $q$-cones of two equivalent $GF(q)$-representations of a matroid $M$ are isomorphic. The question that he asks is whether two inequivalent $GF(q)$-representations of $M$ must always produce isomorphic $q$-cones. We answer this question negatively in the next section. Because, as described above, the rank-2 case is special, we give two examples, one when $M$ has rank 2, and a second when $M$ has rank 3.

2. The Examples

Let $M_\alpha$ be a 4-point line represented over $GF(9)$ by the matrix

$$
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & \alpha \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

where $\alpha$ in $GF(9) - \{0\}$. Let $p = (0,0,1)^T$ and let $M'_\alpha$ be the 9-cone of $M_\alpha$ having base $E(M_\alpha)$ and apex $p$.

**Theorem 2.1.** For all $\alpha \in GF(9) - GF(3)$, the matroids $M'_\alpha$ and $M'_{\alpha^{-1}}$ are non-isomorphic 9-cones of a 4-point line.

**Proof.** Assume the contrary. Since $p$ is the unique point in each of $M'_\alpha$ and $M'_{\alpha^{-1}}$ lying on four 10-point lines, the isomorphism between $M'_\alpha$ and $M'_{\alpha^{-1}}$ must map $p$ to $p$. Clearly $M'_{\alpha^{-1}}$ has a restriction that is isomorphic to $PG(2, 3)$ and uses $p$. Hence $M'_\alpha$ has a restriction $N$ that is isomorphic to $PG(2, 3)$ and uses $p$.

The 12 points of $E(N) - \{p\}$ lie, three to a line, on the four lines $L_1, L_2, L_3,$ and $L_4$ through $p$ and each of $(0,1,0)^T,$ $(1,0,0)^T,$ $(1,1,0)^T,$ and $(1,\alpha,0)^T,$ respectively. Let $(1,\alpha,a)^T$ be a point of $N$ from $L_4,$ and let $(0,1,b_1)^T,$ $(0,1,b_2)^T,$ and $(0,1,b_3)^T$
ON THE NON-UNIQUENESS OF $q$-CONES OF MATROIDS

JAMES OXLEY AND GEOFF WHITTLE

Abstract. Let $M$ be a rank-$r$ simple matroid that is representable over $GF(q)$. A $q$-cone of $M$ is a matroid $M'$ that is constructed by embedding $M$ in a hyperplane of $PG(r,q)$, adding a point $p$ of $PG(r,q)$ not on $H$, and then adding all the points of $PG(r,q)$ that are on lines joining $p$ to an element of $M$. If $r(M) > 2$ and $M$ is uniquely representable over $GF(q)$, then $M'$ is unique up to isomorphism. This note settles a question made explicit by Kung by showing that if $r(M) = 2$ or if $M$ is not uniquely representable over $GF(q)$, then $M'$ need not be unique.

1. Introduction

The matroid terminology used here will follow Oxley [5] with one exception that will be discussed in detail beginning in the third paragraph. The construction, described in the abstract, of a $q$-cone of a simple $GF(q)$-representable matroid $M$ is a natural one. It was introduced by Whittle [6], who called the construction a $q$-lift. He showed that every $q$-cone of a tangential $k$-block over $GF(q)$ is a tangential $(k+1)$-block. The operation also appears in Kung [3, p.36] where it is called framing. Implicit in Whittle's paper is the question of whether non-isomorphic matroids can arise as $q$-cones of the same matroid $M$. This problem is made explicit by Kung [4, p.103]. The purpose of this note is to solve this problem.

If $M$ is a rank-$r$ simple $GF(q)$-representable matroid, then $M'$ is a $q$-cone of $M$ with base $E$ and apex $p$ if the following conditions hold:

(i) $E$ is a set of points of $PG(r,q)$ such that $M \cong PG(r,q)|E$;
(ii) $p$ is a point of $PG(r,q)$ that is not contained in the subspace of $PG(r,q)$ spanned by $E$; and
(iii) the elements of $M'$ are all of the points of $PG(r,q)$ that lie on lines through $p$ and some element of $E$.

Kung [4, p.102] notes that it is easy to see that, for fixed $E$, altering the choice of $p$ subject to (ii) produces a matroid isomorphic to $M'$. One can also change $E$ and still obtain a matroid isomorphic to $M'$ but, to state this observation more precisely, we shall need to discuss equivalent representations of matroids. Our discussion is somewhat extended since we wish to clarify the relationship between several notions of equivalence in the literature. Let $M$ be a rank-$r$ matroid on the set $\{e_1, e_2, \ldots, e_n\}$ where $r \geq 1$. Let $A_1$ and $A_2$ be $r \times n$ matrices over $GF(q)$ with the columns of each being labelled, in order, by $e_1, e_2, \ldots, e_n$. Assume that the identity map on $\{e_1, e_2, \ldots, e_n\}$ is an isomorphism between $M$ and $M[A_i]$ for each $i$ in $\{1, 2\}$. We define $A_1$ and $A_2$ to be algebraically equivalent $GF(q)$-representations of $M$ if $A_2$ can be obtained from $A_1$ by a sequence of operations each consisting of an elementary row operation, a column scaling, or the replacement of each matrix entry