DRAWING SUBDIVISIONS OF COMPLETE AND COMPLETE
BIPARTITE GRAPHS ON BOOKS

ROBIN BLANKENSHIP AND BOGDAN OPOROWSKI

ABSTRACT. We investigate book-thickness of subdivided complete and subdivided complete bipartite graphs. We discuss well-known results that the book-thickness of each of $K_n$ and $K_{n,n}$ is large when $n$ is large, while, for every $n$, some subdivision of $K_n$ and some subdivision of $K_{n,n}$ have book-thickness at most three. The main result of this paper, whose proof is based on Ramsey theory, states that every graph obtained from $K_n$ and $K_{n,n}$ by subdividing each edge at most once has large book-thickness when $n$ is large. Some generalizations of this result are also discussed.

1. INTRODUCTION

Graph theory is a very youthful and vibrant part of mathematics. Many of its problems and results are readily accessible to a general audience. One of its particularly attractive areas, topological graph theory, deals with embedding graphs, viewed as topological spaces, into other topological spaces. In this paper, we will focus on embedding particular kinds of graphs: subdivisions of complete graphs and subdivisions of complete bipartite graphs, into a particular kind of topological space, namely, a book.

We will consider graphs that are finite and have no loops and no multiple edges. For basic graph terminology, see [2]. A graph $G$ can be embedded in a topological space $X$ if the vertices of $G$ can be represented by distinct elements of $X$ and each edge of $G$ can be represented by an arc in $X$, that is, the image of a 1–1 continuous function from the unit interval $[0, 1]$ into $X$. Moreover, the endpoints of every such arc correspond to the endvertices of the edge represented by the arc, and the interior of every such arc avoids all other arcs and all points representing the vertices.

Much of the attention of this paper will focus on complete and complete bipartite graphs. In $K_n$, the complete graph on $n$ vertices, every two vertices are adjacent. In the complete bipartite graph $K_{m,n}$, the vertex set is partitioned into subsets of size $m$ and $n$, and two vertices are adjacent exactly when they are in different such subsets.

To subdivide a graph $K$, we replace each edge of $K$ by a non-zero length path with the same endpoints as the edge it replaces. The class of all graphs that are subdivisions of $K$ will be denoted by $\text{Sub}(K)$. By $\text{Sub}_n(K)$, we will denote the class of graphs obtained from $K$ by replacing each of its edges by a path with at most $n$ internal vertices.

Date: May 25, 1990.

1991 Mathematics Subject Classification. Primary: 05C10; Secondary: 05C55.

Key words and phrases. Complete graph, complete bipartite graph, embedding, book-thickness, Ramsey theory.
A book consists of a non-empty collection of half-planes, called pages, the boundaries of which are identified along a line, called the spine. For a given graph $G$, a natural problem is to determine the least integer $n$ such that $G$ can be embedded in, that is, drawn without crossings on, the $n$-page book. Clearly, a graph can be embedded in a 1-page book if and only if it can be embedded in a 2-page book. The latter occurs exactly when the graph is planar. The following classical theorem of Kuratowski [6] gives an elegant characterization of such graphs.

**Theorem 1.1.** A graph is planar if and only if none of its subgraphs is isomorphic to a subdivision of $K_5$ or of $K_{3,3}$.

The following is a surprising result of Atenos [1].

**Theorem 1.2.** Every graph can be embedded in a 3-page book.

*Proof.* Let $G$ be a graph. Represent all vertices of $G$ on page 1 of the 3-page book and connect every vertex $v$ by straight line segments (as many as $v$ has neighbors in $G$) on page 1 to the spine. Now, to complete the embedding, one only has to connect appropriate pairs of points of the spine by pairwise-disjoint arcs using pages 2 and 3. This is clearly possible since the union of pages 2 and 3 forms a plane. □

The last theorem means that arbitrary book embeddings are somewhat uninteresting. Motivated by such practical applications as VLSI design [3], we now focus on a particular type of book embedding. Specifically, we require that every vertex of $G$ be embedded on the spine, and that the interior of every edge lie on a single page. The least number of pages needed to embed a graph $K$ in a book subject to these restrictions, is called the book-thickness of $K$ and is denoted by $BT(K)$. These restrictions profoundly change the nature of embedding graphs in books. Whereas, by Theorem 1.2, $K_n$ can be embedded in a 3-page book, we show in the next section that $BT(K_n) \geq \lceil \frac{n}{2} \rceil$. By contrast, we can modify the proof of Theorem 1.2 to show that, for all $n$, there is a subdivision of $K_n$ whose book-thickness is at most three. We show in this paper that if every edge of $K_n$ is subdivided at most once, then the book-thickness of the resulting graph is large when $n$ is large. In particular, there is no fixed bound $p$, independent of $n$, such that every graph in $\text{Sub}_1(K_n)$ has book-thickness at most $p$. This result follows from Theorem 3.1, the main result of the paper. In fact, Theorem 3.1 can be strengthened to the following result, whose proof, although based on ideas similar to those of the proof of Theorem 3.1, is too complicated to be included in this paper.

**Theorem 1.3.** For every pair of positive integers $p$ and $m$, there is an integer $n$ such that $BT(G) \geq p$ for every $G \in \text{Sub}_m(K_N) \cup \text{Sub}_m(K_{N,N})$ and every $N \geq n$. □

We conjecture that Theorem 1.3 can be generalized further as follows.

**Conjecture 1.4.** For every pair of positive integers $p$ and $m$, there is an integer $n$ such that if $K$ is a graph of book-thickness at least $n$, then $BT(G) \geq p$ for every graph $G$ in $\text{Sub}_m(K)$. □

2. PRELIMINARY RESULTS

Graphs of book-thickness one and two are easy to characterize. We state these characterizations below and invite the reader to supply proofs.
Theorem 2.1. A graph has book-thickness one if and only if it is outerplanar, that is, if it can be drawn on the plane with all of its vertices incident with the infinite face.

We note that outerplanar graphs can also be characterized as those graphs that do not have subgraphs isomorphic to subdivisions of either \( K_4 \) or \( K_{2,3} \).

Theorem 2.2. A graph has book-thickness at most two if and only if it is a subgraph of a planar hamiltonian graph, that is, a planar graph having a cycle through all of its vertices.

For \( n \) exceeding two, no good characterization of graphs of book-thickness \( n \) is known. There are, however, many results on upper and lower bounds on book-thickness of certain types of graphs. One of the simplest of these is the following well-known result.

Theorem 2.3. For every positive integer \( n \), \( \text{BT}(K_n) \geq \lceil \frac{n}{2} \rceil \), and \( \text{BT}(K_{n,n}) \geq \lceil \frac{n}{2} \rceil \).

Proof. We prove the inequality for \( K_n \), and leave the case of \( K_{n,n} \) to the reader. The assertion is trivial for \( n \) in \( \{1, 2, 3\} \), and so we may assume that \( n \geq 4 \). Consider embedding \( K_n \) in a book with the fewest possible number \( p \) of pages. Partition the edge set of \( K_n \) into \( p + 1 \) classes as follows: \( A \) consists of those \( n \) edges that join pairs of vertices appearing consecutively on the spine, including the edge joining the two outermost vertices, and, for each \( i \) in \( \{1, 2, \ldots, p\} \), the set \( B_i \) consists of those edges not in \( A \) that are embedded in page \( i \). Clearly, for each \( i \), the subgraph of \( G \) induced by \( A \cup B_i \) is outerplanar with the edges in \( A \) forming the outer cycle and the edges in \( B_i \) embedded inside this cycle. An easy inductive argument shows that \( |B_i| \leq n - 3 \). Consequently, \( |E(K_n)| \leq n + p(n - 3) \). On the other hand, \( |E(K_n)| = \binom{n}{2} \). Combining the last two observations, we get

\[
p(n - 3) \geq \binom{n}{2} - n = \frac{n(n - 1) - 2n}{2} = \frac{n(n - 3)}{2}.
\]

Hence, \( p \geq \frac{n}{2} \), and, since \( p \) is an integer, \( p \geq \lceil \frac{n}{2} \rceil \).

We remark that the bound on \( \text{BT}(K_n) \) in Theorem 2.3 is best possible, in fact, it is shown in [3] that \( \text{BT}(K_n) = \lceil \frac{n}{2} \rceil \). On the other hand, the bound on \( \text{BT}(K_{n,n}) \) is not best possible, for example, \( \text{BT}(K_{3,3}) = 3 \neq \lceil \frac{3}{2} \rceil \).

The book-thickness of a graph can be bounded from above in terms of the simplest surface in which the graph can be embedded. In particular, Yamakakis [9] proved the following:

Theorem 2.4. Every planar graph has book-thickness at most four.

The bound in Theorem 2.4 is best possible—there are planar graphs whose book-thickness is four. However, the construction of such graphs is quite difficult (see [9]). The reader is invited to find a planar graph whose book-thickness exceeds two.

Endo [4] proved the following:

Theorem 2.5. Every graph that embeds in a torus has book-thickness at most seven.

For similar, more general results, see [5] and [7].
3. Main Result

This section contains the statement and proof of the main result of the paper. Roughly speaking, this result states that, for large $n$, the graphs obtained from $K_n$ or $K_{n,n}$ by subdividing each edge at most once have large book-thickness. The precise formulation follows:

**Theorem 3.1.** For every integer $p$, there is an integer $n$ such that $\text{BT}(G) > p$ for every $G$ in $\text{Sub}_1(K_N) \cup \text{Sub}_1(K_{N,N})$ and every $N \geq n$.

Before proving Theorem 3.1, we state one of the classical results in combinatorics, Ramsey’s Theorem [8]. Among the many versions of Ramsey’s Theorem, we choose the one which will be the most useful in our proof, and which guarantees that, for sufficiently large $N$, if the edges of $K_N$ or $K_{N,N}$ are colored using a small number of colors, then there is a monochromatic subgraph isomorphic to $K_n$ or $K_{n,n}$, respectively.

**Theorem 3.2.** For any pair of positive integers $c$ and $s$, there are integers $\rho_1(c,s)$ and $\rho_2(c,s)$ such that, for every $n \geq \max\{\rho_1(c,s),\rho_2(c,s)\}$, if each of the edges of $K_n$ and of $K_{n,n}$ is colored by one of the available $c$ colors, then $K_n$ has a monochromatic subgraph isomorphic to $K_s$, and $K_{n,n}$ has a monochromatic subgraph isomorphic to $K_{s,s}$.

**Proof of Theorem 3.1.** Suppose the theorem fails, that is, there is an integer $p$ such that for every $n$ there is an integer $N \geq n$ and a graph $G$ in $\text{Sub}_1(K_N) \cup \text{Sub}_1(K_{N,N})$ such that $\text{BT}(G) \leq p$. Clearly, $p > 1$. We will only consider the case when $G$ is in $\text{Sub}_1(K_N)$, leaving the proof in the case when $G$ is in $\text{Sub}_1(K_{N,N})$ to the reader. Let $n = \rho_1(\frac{N}{5}, 5)$, where $\rho_1$ is the function described in Theorem 3.2. Embed $G$, which is in $\text{Sub}_1(K_N)$, in a $p$-page book whose pages have been numbered 1, 2, ..., $p$. Take a separate copy of $K_N$ and consider a natural bijection between the edges of the $K_N$ and the one- and two-edge paths of $G$. We will color the edges of $K_N$ with colors determined by the book embedding of $G$. Specifically, for every edge $e$, the corresponding path of $G$, having at most two edges, is embedded in the union of two distinct pages, say $i$ and $j$. (If the path is embedded in just one page, then the other page may be chosen arbitrarily.) Consider the pair $\{i, j\}$ as a color and assign it to $e$. Consequently, the edges of $K_N$ have been colored with $\binom{p}{2}$ colors. Upon applying Theorem 3.2 to the edge-colored $K_N$, we conclude that it contains a monochromatic complete subgraph $K$ on five vertices. The subgraph $H$ of $G$ that corresponds to $K$ is a subdivision of $K_5$ and it is embedded in just two pages of the book. This is a contradiction, since, by Theorem 1.1, $H$ is nonplanar and so, by Theorem 2.2, cannot be embedded in two pages.

**References**


(Robin Blankenship and Bogdan Oporowski) Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70863-4018, USA

E-mail address: [robin, bogdan]@math.lsu.edu