FOCK SPACES CORRESPONDING TO POSITIVE DEFINITE LINEAR TRANSFORMATIONS

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Abstract. Suppose $A$ is a positive real linear transformation on a finite dimensional complex inner product space $V$. The reproducing kernel for the Fock space of square integrable holomorphic functions on $V$ relative to the ‘Gaussian’ measure $d\mu_A(z) = \frac{1}{(2\pi)^{d/2}} e^{-\|z\|^2/2} dz$ is described in terms of the holomorphic-antiholomorphic decomposition of the linear operator $A$. Moreover, if $A$ commutes with a conjugation on $V$, then a restriction mapping to the real vectors in $V$ is polarized to obtain a Segal-Bargmann transform.

1. Introduction

The classical Segal-Bargmann transform is an integral transform which defines an unitary isomorphism of $L^2(\mathbb{R}^n)$ onto the Hilbert space $F(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$ which are square integrable with respect to the Gaussian measure $\mu = \pi^{-n} e^{-\|z\|^2} dxdy$, where $dxdy$ stands for the Lebesgue measure on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, see [1, 2, 3, 4, 8, 9]. There have been several generalizations of this transform, based on the heat equation or the representation theory of Lie groups [5, 7, 10]. In particular, it was shown in [7] that the Segal-Bargmann transform is a special case of the restriction principle, i.e., construction of unitary isomorphisms based on the polarization of a restriction map. This principle was first introduced in [7], see also [6], where several examples were explained from that point of view. In short the restriction principle can be explained in the following way. Let $M_C$ be a complex manifold and let $M \subset M_C$ be a totally real submanifold. Let $F = F(M_C)$ be a Hilbert space of holomorphic functions on $M_C$ such that the evaluation maps $F \ni F \mapsto F(z) \in \mathbb{C}$ are continuous for all $z \in M_C$, i.e., $F$ is a reproducing Hilbert space. There exists a function $K : M_C \times M_C \rightarrow \mathbb{C}$ holomorphic in the first variable, anti-holomorphic in the second variable, and such that the following holds:

1. $K(z, w) = \overline{K(w, z)}$ for all $z, w \in M_C$;
2. If $K_w(z) := K(z, w)$ then $K_w \in F$ and $F(w) = (F, K_w)$, $\forall F \in F, w \in M_C$.

The function $K$ is the reproducing kernel for the Hilbert space, Let $D : M \rightarrow \mathbb{C}^*$ be measurable. Then the restriction map $RF := DF|_M$ is injective. Assume that

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there is a measure \( \mu \) on \( M \) such that \( RF \in L^2(M, \mu) \) for all \( F \) in a dense subset of \( F \). Provided \( R \) is closeable, polarizing \( R^* \) we can write

\[
R^* = U|R^*|
\]

where \( U : L^2(M, \mu) \to F \) is an unitary isomorphism. Using that \( F \) is a reproducing Hilbert space we get that

\[
Uf(z) = (Uf, K_z) = (f, U^*K_z) = \int_M f(m)\overline{(U^*K_z)(m)} \, d\mu(m).
\]

Thus \( Uf \) is always an integral operator. We notice also that the formula for \( U \) shows that the important object in this analysis is the reproducing kernel \( K(z, w) \). The reproducing kernel for the classical Fock space is given by \( K(z, w) = e^{-|z-w|^2} \). By taking \( D(z) := (2\pi)^{-n/4}e^{-\|z\|^2} \), which is closely related to the heat kernel, we arrive at the classical Segal–Bargmann transform

\[
Ug(z) = (2/\pi)^{n/4}e^{-(z, z)/2} \int g(y)e^{-(z-y, z-y)} \, dy
\]

where \( (z, w) = \sum z_iw_i \). The same principle can be used to construct the Hall–transform for compact Lie groups, [5]. In [10] A. Sen Gupta introduced a Fock space and Segal–Bargmann transform depending on two parameters \( r, s > 0 \), giving different weights to the \( x \) and \( y \) directions, where \( z = x + iy \in \mathbb{C}^n \). Thus \( F \) is now the space of holomorphic functions \( F(z) \) on \( \mathbb{C}^n \) which are square-integrable with respect to the Gaussian measure \( dM_{r,s}(z) = \frac{1}{(\pi r)^{n/2} s^{n/2}} e^{-\frac{1}{r^2} - \frac{z^2}{s^2}} \). In [10] the reproducing kernel and the Segal–Bargmann transform for this space is worked out. This construction has a natural generalization by viewing \( r^{-1} \) and \( s^{-1} \) as the diagonal elements in a positive definite matrix \( A = d(r^{-1}I_n, s^{-1}I_n) \). The measure is then simply

\[
(1.1) \quad dM_{r,s}(z) = \frac{\sqrt{\det(A)}}{\pi^n} e^{-\|z\|^2} \, dxdy
\]

and this has meaning for any positive definite matrix \( A \).

In this paper we show that (1.1) gives rise to a Fock space \( F_A \) for arbitrary positive matrices \( A \). We find an expression for the reproducing kernel \( K_A(z, w) \). We use the restriction principle to construct a natural generalization of the Segal–Bargmann transform for this space.

2. The Fock space and the restriction principle

In this section we recall some standard facts about the classical Fock space of holomorphic function on \( \mathbb{C}^n \). We refer to [4] for details and further information. Let \( \mu \) be the measure \( d\mu = \pi^{-n} e^{-\|z\|^2} \, dxdy \) and let \( F \) be the classical Fock-space of holomorphic functions \( F : \mathbb{C}^n \to \mathbb{C} \) such that

\[
\|F\|^2 := \int |F(z)|^2 \, d\mu(z) < \infty.
\]

The space \( F \) is a reproducing Hilbert space with inner product

\[
(F, G) = \int F(z)\overline{G(z)} \, d\mu
\]
and reproducing kernel $K(z, w) = e^{z \cdot w}$ where $z \cdot w = z_1 w_1 + \cdots + z_n w_n$. Thus

$$F(w) = \int F(z) \overline{K(z, w)} \, d\mu = (F, K_w)$$

where $K_w(z) = K(z, w)$. The function $K(z, w)$ is holomorphic in the first variable, anti-holomorphic in the second variable, and $K(z, w) = \overline{K(w, z)}$. Notice that $K(z, z) = (K_z, K_z)$. Hence $\|K_z\| = e^{\|z\|^2}$. Finally the linear space of finite linear combinations $\sum c_j K_{z_j}$, $z_j \in \mathbb{C}^n$, $c_j \in \mathbb{C}$, is dense in $F$. An orthonormal system in $F$ is given by the monomials $e_\alpha(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}/\sqrt{\alpha_1! \cdots \alpha_n!}$, $\alpha \in \mathbb{N}_0^n$.

View $\mathbb{R}^n \subset \mathbb{C}^n$ as a totally real submanifold of $\mathbb{C}^n$. We will now recall the construction of the classical Segal-Bargmann transform using the restriction principle, see [6, 7]. For constructing a restriction map as explained in the introduction we need to choose the function $D(x)$. One motivation for the choice of $D$ is the heat kernel, but another one, more closely related to representation theory, is that the restriction map should commute with the action of $\mathbb{R}^n$ on the Fock space and $L^2(\mathbb{R}^n)$. Indeed, take

$$T(x)F(z) = M(x, z)F(z - x)$$

for $F$ in $F$ where $M(x, z)$ has properties sufficient to make $x \mapsto T(x)$ a unitary representation of $\mathbb{R}^n$ on $F$. Namely, $M$ is a multiplier, i.e., $M(x, z)M(y, z - x) = M(x + y, z)$; $z \mapsto M(x, z)$ is holomorphic in $z$ for each $x$; and $|M(x, z)| = (\frac{dy(z - x)}{dy(z)})^\frac{1}{2} = e^{\text{Re}z - \|x\|^2/2}$. Note $M(x, z) := e^{z \cdot x - \|x\|^2/2}$ has these properties.

Set $D(x) = (2\pi)^{-n/4}M(0, x) = (2\pi)^{-n/4}e^{-\|x\|^2/2}$ and define $R : F \to C^\infty(\mathbb{R}^n)$ by

$$RF(x) := D(x)F(x) = (2\pi)^{-n/4}e^{-\|x\|^2/2}F(x).$$

Then

$$RT(y)F(x) = (2\pi)^{-n/4}e^{-\|y\|^2/2}T(y)F(x)$$

$$= (2\pi)^{-n/4}e^{-\|x - y\|^2/2}e^{-\|y\|^2/2}F(x - y)$$

$$= (2\pi)^{-n/4}e^{-\|x - y\|^2/2}F(x - y)$$

$$= RF(x - y).$$

As $\mathbb{R}^n$ is a totally real submanifold of $\mathbb{C}^n$, it follows that $R$ is injective. Furthermore the holomorphic polynomials $p(z) = \sum a_\alpha z^\alpha$ are dense in $F$ and obviously $Rp \in L^2(\mathbb{R}^n)$, Hence all the Hermite functions $h_\alpha(x) = (-1)^{\|\alpha\|}(D^{\alpha}e^{-\|x\|^2})e^{x\cdot y}/\sqrt{\alpha_1! \cdots \alpha_n!}$ are in the image of $R$; so $\text{Im}(R)$ is dense in $L^2(\mathbb{R}^n)$ and $R$ is a densely defined operator from $F$ into $L^2(\mathbb{R}^n)$. It follows easily from the fact that the maps $F \mapsto F(z)$ are continuous, that $R$ is a closed operator, Hence $R$ has an adjoint $R^* : L^2(\mathbb{R}^n) \to F$. For $z, w \in \mathbb{C}^n$, let $(z, w) = \sum z_j w_j$. Then:

$$R^* g(z) = (R^* g, K_z)$$

$$= (g, RK_z)$$

$$= (2\pi)^{-n/4} \int g(y)e^{-\|y\|^2/2}e^{z \cdot y} \, dy$$

$$= (2\pi)^{-n/4}e^{(z, z)/2} \int g(y)e^{-\|z - y, z - y\|^2/2} \, dy$$

$$= (2\pi)^{n/4}e^{(z, z)/2} g * p(z)$$
where \( p(z) = (2\pi)^{-n/2}e^{-(z, z)/2} \) is holomorphic. Hence
\[
(2.1) \quad RR^* g(x) = g * p(x).
\]
As \( p \in L^1(\mathbb{R}^n) \), it follows that
\[
\|RR^*\| \leq \|p\|_1; \text{ so } RR^* \text{ is continuous.}
\]

\[(R^* g, R^* g) = (RR^* g, g) \leq \|RR^*\| \|g\|_2.
\]

Thus

**Lemma 2.1.** The maps \( R \) and \( R^* \) are continuous.

Let \( p_t(x) = (2\pi t)^{-n/2}e^{-(x,x)/2t} \) be the heat kernel on \( \mathbb{R}^n \). Then \((p_t)_{t>0}\) is a convolution semigroup and \( p = p_1 \). Hence \( \sqrt{RR^*} = p_{1/2} \star \) or

\[
RUg(x) = |R^*| g(x) = p_{1/2} \ast g(x) = \pi^{-n/2} \int g(y)e^{-(x-y,x-y)}dy.
\]

It follows that

\[
Ug(z) = (2/\pi)^{n/4} \int g(y) \exp(-(y,y) + 2(z,y)-(z,z)/2) dy
\]
for \( z \in \mathbb{R}^n \). But the function on the right hand side is holomorphic in \( z \). Analytic continuation gives the following theorem.

**Theorem 2.2.** The map \( U : L^2(\mathbb{R}^n) \to F \) given by

\[
Ug(z) = (2/\pi)^{n/4} \int g(y) \exp(-(y,y) + 2(z,y)-(z,z)/2) dy
\]
is an unitary isomorphism. \( U \) is called the Segal–Bargmann transform.

3. Twisted Fock spaces

Let \( V \simeq \mathbb{C}^n \) be a finite dimensional complex vector space of complex dimension \( n \) and let \( \langle \cdot, \cdot \rangle \) be a complex inner product. As before we will sometimes write \( \langle z, w \rangle = z \cdot \bar{w} \). We will also consider \( V \) as a real vector space with real inner product defined by \( \langle z, w \rangle = \text{Re} \langle z, w \rangle \). Notice that \( \langle z, z \rangle = \langle z, z \rangle \) for all \( z \in \mathbb{C}^n \). Let \( J \) be the real linear transformation of \( V \) given by \( Jz = i \bar{z} \). Note that \( J^* = -J = J^{-1} \) and thus \( J \) is a skew symmetric real linear transformation. Fix a real linear transformation \( A \). Then \( A = H + K \) where

\[
H := \frac{A + J^{-1}AJ}{2} \quad \text{and} \quad K := \frac{A - J^{-1}AJ}{2}.
\]

Note that \( HJ = \frac{1}{2}(AJ - J^{-1}A) = \frac{1}{2}J(J^{-1}AJ + A) = JH \) and \( KJ = \frac{1}{2}(AJ + J^{-1}A) = \frac{1}{2}J(J^{-1}AJ - A) = -JK \). Furthermore \( H \) is complex linear and \( K \) is conjugate linear. We assume that \( A \) is positive definite.

**Lemma 3.1.** The complex linear transformation \( H \) is self adjoint, positive with respect to the inner product \( \langle \cdot, \cdot \rangle \), and invertible.

**Proof.** Since \( A \) is positive and invertible as a real linear transformation, we have \( \langle Az, z \rangle > 0 \) for all \( z \neq 0 \). But \( J \) is real linear and skew symmetric, hence \( \langle JAJ^{-1}z, z \rangle > 0 \) for all \( z \neq 0 \). In particular \( H = \frac{1}{2}(A + JAJ^{-1}) \) is complex linear, symmetric with respect to the real inner product \( \langle \cdot, \cdot \rangle \), and positive. We know \( \langle Hv, w \rangle = \text{Re} \langle v, Hw \rangle \). Thus \( \text{Re} \langle Hv, w \rangle = \text{Re} \langle v, Hw \rangle \). From this we obtain

\[
\text{Re} \langle Hv, w \rangle = \text{Re} \langle iv, Hw \rangle.
\]
This implies \( \text{Im} \langle Hv, w \rangle = \text{Im} \langle v, Hw \rangle \). Putting these together gives \( \langle Hv, w \rangle = \langle v, Hw \rangle \). Hence \( H \) is complex self adjoint and \( \langle Hz, z \rangle > 0 \) for \( z \neq 0 \). \( \square \)

**Lemma 3.2.** Let \( w \in V \). Then \( \langle Aw, w \rangle = (Aw, w) + i \text{Im}(Kw, w) \) and \( (Aw, w) = (Hw, w) + (Kw, w) \).

**Proof.** Let \( w \in V \). Then

\[
\langle Aw, w \rangle = (Hw, w) + (Kw, w)
\]

\[
= (Hw, w) + (Kw, w) + i \text{Im}(Kw, w)
\]

\[
= (Aw, w) + i \text{Im}(Kw, w).
\]

This implies the first statement. Taking the real part in the second line gives the second claim. \( \square \)

Denote by \( \det_V \) the determinant of a \( \mathbb{R} \)-linear map on \( \mathbb{C}^n \simeq \mathbb{R}^{2n} \). Let \( d\mu_A(z) = \pi^{-n} \sqrt{\det_V A} e^{-(Az, z)} dx dy \) and let \( F_A \) be the space of holomorphic functions \( F : \mathbb{C}^n \to \mathbb{C} \) such that

\[
\|F\|_A^2 := \int |F(z)|^2 d\mu_A < \infty.
\]

Our normalization of \( d\mu \) is chosen so that \( ||1||_A = 1 \). Just as in the classical case one can show that \( F_A \) is a reproducing Hilbert space, but this will also follow from the following Lemma. We notice that all the holomorphic polynomials \( p(z) \) are in \( F \). To simplify the notation, we let \( T = H^{-1/2} \). Then \( T \) is symmetric, positive definite and complex linear. Let \( c_A = \sqrt{\det_V (A^{1/2}T)} = (\det_V (A)/\det_V (H))^{1/4} \).

**Lemma 3.3.** Let \( F : V \to \mathbb{C} \) be holomorphic. Then \( F \in F_A \) if and only if \( F \circ T \in F \) and the map

\[
\Psi(F)(w) := c_A \exp \left( -\langle Kw, Tw \rangle /2 \right) F(Tw)
\]

is an unitary isomorphism. In particular

\[
\Psi^* F(w) = \Psi^{-1} F(w) = c_A^{-1} \exp \left( \langle Kw, w \rangle /2 \right) F(\sqrt{H}w).
\]

**Proof.** Let \( F : V \to \mathbb{C} \). Then \( F \) is holomorphic if and only if \( F \circ T \) is holomorphic as \( T \) is complex linear and regular. We also have:

\[
\|\Psi F\|^2 = \pi^{-n} \int |\Psi F(w)|^2 e^{-\langle w, w \rangle} dw
\]

\[
= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-\langle Kw, w \rangle} e^{-(\sqrt{H}, \sqrt{H})} dw
\]

\[
= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-\langle Kw, w \rangle} e^{-(H, w)} dw
\]

\[
= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-\langle (H+K)w, w \rangle} dw
\]

\[
= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-\langle K, w \rangle} dw
\]

\[
= \|F\|_A^2
\]

and thus, by polarization, \( \Psi \) is unitary. \( \square \)

**Theorem 3.4.** The space \( F_A \) is a reproducing Hilbert space with reproducing kernel

\[
K_A(z, w) = c_A^{-2} e^{\frac{1}{2} \langle Hz, z \rangle} e^{\frac{1}{2} \langle Kw, w \rangle}.
\]
Proof. By Lemma 3.3 we get
\[ c_A \exp(-\langle KTw, Tw \rangle /2) F(Tw) = \Psi(F)(w) = (\Psi(F), K_w) = (F, \Psi^*(K_w)). \]

Hence
\[ K_A(z, w) = c_A^{-1} \exp(\langle KTw, Tw \rangle /2) \Psi^*(K \sqrt{T}w) = c_A^{-2} e^{\frac{1}{2} \langle HZ, w \rangle} e^{\frac{1}{2} \langle Kw, w \rangle}. \]

\[ \square \]

4. The Restriction Map

We assume as before that A > 0. We notice that Lemma 3.3 gives a unitary isomorphism \( \Psi^*U : L^2(\mathbb{R}^n) \to F_A \), where U is the classical Segal-Bargmann transform. But this is not the natural transform that we are looking for. As H is positive definite there is an orthonormal basis \( e_1, \ldots, e_n \) of V and positive numbers \( \lambda_j > 0 \) such that \( He_j = \lambda_j e_j \). Let \( V_\mathbb{R} := \sum \mathbb{R}e_j \). Set \( \sigma(\sum a_i e_i) = \sum \bar{a}_i e_i \). Then \( \sigma \) is a conjugation with \( V_\mathbb{R} = \{ z : \sigma z = z \} \). We say that a vector is real if it belongs to \( V_\mathbb{R} \). As \( He_j = \lambda_j e_j \) with \( \lambda_j \in \mathbb{R} \) it follows that \( HV_\mathbb{R} \subseteq V_\mathbb{R} \). We denote by det the determinant of a \( \mathbb{R} \)-linear map of \( V_\mathbb{R} \).

Lemma 4.1. \( \langle Kz, w \rangle = \langle Kw, z \rangle \).

Proof. Note that \( \sigma K \) is complex linear. Since \( J^* = -J, K = \frac{1}{2}(A - JAJ^{-1}) \) is real symmetric. Thus \( \langle Kw, z \rangle = \langle w, Kz \rangle = \langle Kz, w \rangle \). Also note \( \langle iKz, w \rangle = \langle JKz, w \rangle = -(Jz, Kw) = -\langle z, Kw \rangle \). Hence \( \text{Re}(iKz, w) = -\text{Re}(iz, Kw) \). So \( -\text{Im}(Kz, w) = \text{Im}(z, Kw) \). This gives \( \text{Im}(Kw, z) = \text{Im}(Kz, w) \). Hence \( \langle Kz, w \rangle = \langle Kw, z \rangle \). \( \square \)

Lemma 4.2. \( \langle \sigma K \rangle^* = K \sigma \).

Proof. We have \( \langle \sigma z, w \rangle = \langle w, \sigma z \rangle \). Hence
\[ \langle \sigma Kz, w \rangle = \langle \sigma w, \sigma^2 Kz \rangle = \langle \sigma w, Kz \rangle = \langle z, K \sigma w \rangle. \] \( \square \)

Corollary 4.3. If \( x, y \in V_\mathbb{R} \), then \( \langle Hx, y \rangle \) is real.

Proof. Clearly \( \langle \cdot, \cdot \rangle \) is real on \( V_\mathbb{R} \times V_\mathbb{R} \). Since \( HV_\mathbb{R} \subseteq V_\mathbb{R} \), we see \( \langle Hx, y \rangle \) is real. \( \square \)

Lemma 4.4. Define \( m : V_\mathbb{R} \times V \to \mathbb{C} \) by \( m(x, z) = e^{\langle Hx, z \rangle} e^{\langle Kx, x \rangle} e^{-(Ax, x)/2} e^{-(H(z-x), y)} e^{-(K(z-x), y)} e^{-(A\bar{y}, \bar{y})}/2 \).

Then \( m \) is a multiplier. Moreover, if \( T_xF(z) := m(x, z)F(z-x) \), then \( T \) is a representation of the abelian group \( V_\mathbb{R} \) on \( F_A \). It is unitary if \( KV_\mathbb{R} \subseteq V_\mathbb{R} \).

Proof. We first show \( m \) is a multiplier.

\[ m(x, z)m(y, z-x) = e^{\langle Hx, z \rangle} e^{\langle Kx, x \rangle} e^{-(Ax, x)/2} e^{-(H(z-x), y)} e^{-(K(z-x), y)} e^{-(A\bar{y}, \bar{y})}/2 \]
\[ = e^{\langle Hx, y \rangle} e^{\langle Kx, x \rangle} e^{-(Ax, x)/2} e^{-(H(z-x), y)} e^{-(K(z-x), y)} e^{-(A\bar{y}, \bar{y})}/2 \]
\[ = e^{\langle Hx, x-y \rangle} e^{\langle Kx, x-y \rangle} e^{-(Ax, x)/2} e^{-(H(x-y), y)} e^{-(K(x-y), y)} e^{-(A\bar{y}, \bar{y})}/2 \]
\[ = e^{\langle Hx, x-y \rangle} e^{\langle Kx, x-y \rangle} e^{-(Ax, x-y)/2} e^{-(H(x-y), y)/2} e^{-(K(x-y), y)/2} e^{-(A\bar{y}, \bar{y})}/2 \]
\[ = e^{\langle Hx, x-y \rangle} e^{\langle Kx, x-y \rangle} e^{-(Ax, x-y)/2} \]
\[ = m(z, x+y) \]

for \( \langle Hx, y \rangle \) is real and \( \langle Kx, y \rangle = \langle Ky, x \rangle \) by Lemma 4.1 and Corollary 4.3.
Since m is a multiplier, we have \( T_x T_y = T_{x+y} \). For each \( T_x \) to be unitary, we need \( |m(x, z)| = e^{(Az, x) - (Ax, x)/2} \). But
\[
|m(x, z)| = e^{(Hz, x) e^{(Kz, z)} e^{-(Ax, x)/2}} = e^{(Az, x) - (Ax, x)/2} e^{(Kz, Kz, x)}.
\]
Thus \( T_x \) is unitary for all \( x \) iff the real part of every vector \( Kz - Kz = 0 \). Taking \( z \in \mathbb{V}_R \), we see \( K(iz) - Kiz = K(-iz) + iKz = 2iKz \) must have real part zero, and this is equivalent to \( K \mathbb{V}_R \subseteq \mathbb{V}_R \). □

Notice that \( \det_V H = (\det H)^2 \). To simplify some calculations later on we define \( c := (2\pi)^{-n/4} \left( \frac{\det_v A}{\det H} \right)^{1/4} \). We remark for further reference:

**Lemma 4.5.** \( c_A^{-2} e^2 = \sqrt{\frac{\det H}{(2\pi)^n/2}} \) and \( c^{-1} \sqrt{\frac{\det H}{(2\pi)^n/2}} = \left( \frac{2}{\pi} \right)^{n/4} \left( \frac{\det H}{\det v A} \right)^{1/4} \).

Let \( D(x) = c m(x, 0) = c e^{-i(Ax, x)/2} \) and define \( R : \mathcal{F}_A \to \mathcal{C}^\infty(\mathbb{V}_R) \) by \( RF(x) := D(x)F(x) \). Extending the bilinear form \( x \mapsto (Ax, x) \) to a complex bilinear form \( \langle z, z \rangle_A \) on \( \mathbb{V}_R \) shows that \( D \) has a holomorphic extension to \( \mathbb{V}_R \).

**Lemma 4.6.** The restriction map \( R \) intertwines the action of \( \mathbb{V}_R \) on \( \mathcal{F}_A \) and the left regular action \( L \) on functions on \( \mathbb{V}_R \).

**Proof.** We have
\[
R(T_y F)(x) = c m(x, 0) T_y F(x) = c m(x, 0) m(y, x) F(x - y) = c m(x - y, 0) F(x - y) = L_y RF(x).
\]

□

5. **The Generalized Segal–Bargmann Transform**

As for the classical space, \( R \) is a densely defined, closed operator. It also has dense image in \( L^2(\mathbb{V}_R) \). To see this, let \( \{ h_\alpha \}_\alpha \) be the orthonormal basis of \( L^2(\mathbb{V}_R) \) given by the Hermite functions. Then \( \left\{ \det(A) \frac{1}{2} h_\alpha(\sqrt{Ax}) \right\}_\alpha \) is an orthonormal basis of \( L^2(\mathbb{V}_R) \) which is contained in the image of \( R \). It follows again that \( R \) has an adjoint and
\[
R^* h(z) = (R^* h, K_{A,z}) = (h, RK_{A,z})
\]
where \( K_{A,z}(w) = K_A(w, z) = c_A^{-2} e^{\frac{i}{2} \langle Kw, w \rangle} e^{(H w, z)} e^{\frac{1}{2} (K z, z)} \). Thus
\[
R^* h(z) = c \int h(x) e^{-(Ax, x)/2} K_{A}(x, z) \, dx
= c_A^{-2} c \int h(x) e^{-(Ax, x)/2} e^{\frac{i}{2} \langle K z, z \rangle} e^{(H x, x)} e^{\frac{1}{2} (K x, x)} \, dx
= c_A^{-2} c e^{\frac{i}{2} \langle K z, z \rangle} \int h(x) e^{-(Ax, x)/2} e^{-(K x, x)/2} e^{(H x, x)} e^{\frac{1}{2} (K x, x)} \, dx
= c_A^{-1} c e^{\frac{i}{2} \langle K z, z \rangle} \int h(x) e^{-(Ax, x)/2} e^{(H x, x)} \, dx
= c_A^{-2} c e^{\frac{i}{2} \langle K z, z \rangle} e^{\frac{1}{2} (H z, z)} \int h(x) e^{-(Ax, x) - (H x, x) + (x, H x)}/2 \, dx
= c_A^{-2} c e^{\frac{i}{2} \langle K z, z \rangle} e^{\frac{1}{2} (H z, z)} \int h(x) e^{-|z - x|/2} \, dx.
\]
for \((z, Hx) = (Hx, z) = \langle Hx, z \rangle = \langle z, Hx \rangle\) and \(\langle z, Hx \rangle = \langle z, Hx \rangle\). Thus we finally arrive at

\[
R^* h(z) = e_{-\frac{1}{2}(Ax, x)} e^{\frac{1}{2}(z, Hx + Kz)} e^{-\frac{1}{2}(z, Hx)} * h(z).
\]

Let \(P : \mathbb{V}_R \to \mathbb{V}_R\) be positive. Define \(\phi_P(x) = \sqrt{\det(P)}(2\pi)^{-n/2} e^{-\|Px\|^2/2}\). For \(t > 0\), let \(P(t) = P/t\).

**Lemma 5.1.** Let the notation be as above. Then \(0 < t \mapsto \phi_{P(t)}\) is a convolution semigroup, i.e., \(\phi_{P(t+s)} = \phi_{P(t)} * \phi_{P(s)}\).

**Proof.** This follows by change of parameters \(y = \sqrt{P}x\) from the fact that \(\phi_{I(t)}(x) = (2\pi)^{-n/2} e^{-\|x\|^2/2t}\) is a convolution semigroup. \(\square\)

We define a unitary operator \(W\) on \(L^2(\mathbb{V}_R)\) by

\[
W f(x) = e^{i \text{Im} \langle x, Kx \rangle} f(x) = e^{i \text{Im} \langle x, Ax \rangle} f(x).
\]

We know \(W = I\) if \(KV_R \subseteq \mathbb{V}_R\) and this occurs if \(A\) leaves \(\mathbb{V}_R\) invariant.

**Lemma 5.2.** Let \(h\) be in the domain of definition of \(R^*\). Then \(RR^* h = W(\phi_H * h)\).

**Proof.** We notice first that \(e_{-\frac{1}{2}(Ax, x)} = (2\pi)^{-n/2} \sqrt{\det H}\) by Lemma 4.5. From (5.1) we then get

\[
RR^* h(x) = e^{-\frac{1}{2}(Ax, x)} R^* h(x) = e_{-\frac{1}{2}(Ax, x)} e^{\frac{1}{2}(z, Hx + Kz)} e^{-\frac{1}{2}(z, Hx)} * h(x) = (2\pi)^{-n/2} \sqrt{\det(H)} e^{-\frac{1}{2}(Ax, x)} e^{\frac{1}{2}(z, Ax)} e^{-\frac{1}{2}(z, Hx)} * h(x)
\]

\[
= (2\pi)^{-n/2} \sqrt{\det(H)} e^{i \text{Im} \langle x, Ax \rangle} \int e^{-\frac{1}{2}(y, Hx)} h(x - y) dy.
\]

\[
= (2\pi)^{-n/2} \sqrt{\det(H)} e^{i \text{Im} \langle x, Ax \rangle} \int e^{-\frac{1}{2}(y, Hx)} h(x - y) dy = W(\phi_H * h)(x)
\]

\(\square\)

Lemma 5.1 and Lemma 5.2 leads to the following corollary:

**Corollary 5.3.** Suppose \(AV_R \subseteq \mathbb{V}_R\). Then

\[
|R^* h(x) = \phi_{H(1/2)} * h(x) = \frac{\sqrt{\det(H)}}{\pi^{n/2}} \int_{V_R} e^{-\|y\|^2/2} h(x - y) dy.
\]

**Theorem 5.4** (The Segal–Bargmann Transform). Suppose \(A\) leaves \(\mathbb{V}_R\) invariant. Then the operator \(U_A : L^2(\mathbb{V}_R) \to F_A\) defined by

\[
U_A f(z) = \left(\frac{2}{\pi}\right)^{n/4} \frac{(\det H)^{3/4}}{(\det V A)^{1/4}} e^{\frac{1}{2}(z, z)} e^{\frac{1}{2}(z, z)} \int e^{(H(x, y) + (z, K z))} f(y) dy.
\]

is an unitary isomorphism. The map \(U_A\) is called the generalized Segal–Bargmann transform.
Proof. By polarization we can write $R^* = U |R^*|$ where $U : L^2(V_\mathbb{R}) \to \mathcal{F}_A$ is an
unitary isomorphism. Taking adjoints gives $|R^*| U^* = R$. Hence $RU = |R^*|$. Hence
\[
 c m(x) U h(x) = RU h(x) \\
= \langle |R^*| h(x) \rangle \\
= \sqrt{\frac{\det(H)}{\pi^{n/2}}} \int_{V_n} e^{-\frac{1}{2} \langle (x, H x) + (x, K x) \rangle} h(x - y) \, dy.
\]
Since $m(x) = e^{-\frac{1}{2} \langle (x, H x) + (x, K x) \rangle}$, we have using Lemma 4.5:
\[
U f(x) = \left( \frac{2}{\pi} \right)^{n/4} \frac{1}{\sqrt{\det(V_A)}} e^{\frac{1}{2} \langle (x, H x) + (x, K x) \rangle} \int e^{(x-y, H(x-y))} f(y) \, dy.
\]
By holomorphy, this implies
\[
U f(z) = \left( \frac{2}{\pi} \right)^{n/4} \frac{1}{\sqrt{\det(V_A)}} e^{\frac{1}{2} \langle (H z, z) + (z, K z) \rangle} \int e^{(z-y, H(z-y))} f(y) \, dy
\]
is the Bargmann–Segal transform.

6. The Two Parameter Segal–Bargmann Transform

We assume now that $A = d(r^{-1} I_n, s^{-1} I_n)$ where $t, s > 0$ and $d(\cdot, \cdot)$ stands for
diagonal matrix. Then as $\mathbb{R}$-linear maps on $\mathbb{C}^n$ we have
\[
H = \frac{(r + s)}{2rs} I_{2n} \quad \text{and} \quad K = d(\frac{s - r}{2rs}, \frac{r - s}{2rs}).
\]
Notice that in this case we can take $V_\mathbb{R} = \mathbb{R}^n$ and then $A(V_\mathbb{R}) = V_\mathbb{R}$. We have
\[
T = \sqrt{H^{-1}} = \sqrt{\frac{2rs}{r+s}} I_{2n} \quad \text{and} \quad KT = d(\frac{s-r}{\sqrt{2rs(r+s)}}, \frac{r-s}{\sqrt{2rs(r+s)}}).
\]
Finally we have
\[
\det H = 2^{-n} \left( \frac{r+s}{rs} \right)^n \quad \text{and} \quad \det V_A = (rs)^{-n}.
\]
Thus $c_A = 2^{n/4} (r + s)^{-n/4}$. Hence the following theorem.

Theorem 6.1. Let $r, s > 0$ and let $\mathcal{F}_{r,s}$ be the Fock space of holomorphic functions that are square integrable with respect to the Gaussian measure $dM_{r,s}(z) = \frac{1}{\pi^{(r+s)/2}} e^{-\frac{z^2}{r+s}} \, dz$. Let $\mathcal{F}$ be the classical Fock space. Then the following holds:

1. The map $\Psi_{r,s} : \mathcal{F}_{r,s} \to \mathcal{F}$ given by
\[
\Psi_{r,s}(F)(x) = 2^{n/4} (r + s)^{-n/4} \exp \left( \frac{s-r}{2(r+s)} w^2 \right) F\left( \frac{2rs}{r+s} w \right)
\]
is an unitary isomorphism.

2. The space $\mathcal{F}_{r,s}$ is a reproducing Hilbert space with reproducing kernel
\[
K_{r,s}(z, w) = \left( \frac{r+s}{2} \right)^{n/2} e^{\frac{s-r}{2(r+s)} (|z|^2 + |w|^2)} e^{\frac{s-r}{2(r+s)} z \cdot \bar{w}}.
\]

3. The generalized two parameter Segal–Bargmann transform $U_{r,s} : \mathcal{F}_{r,s} \to L^2(\mathbb{R}^n)$ is given by
\[
U_{r,s} f(z) = \left( \frac{2}{\pi} \right)^{n/4} \left( \frac{r+s}{2} \right)^{3n/4} 1 \frac{1}{(rs)^{n/2}} e^{-z \cdot \bar{w}} \int e^{\frac{s-r}{2(r+s)} (z-y)^2} f(y) \, dy.
\]
REFERENCES


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