GROWTH FUNCTIONS FOR GENERALIZED FUNCTIONS
ON WHITE NOISE SPACE

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ABSTRACT. Let $u$ be a positive continuous function on $[0, \infty)$ satisfying the conditions: (0) $\lim_{r \to \infty} r^{-1/2} \log u(r) = \infty$, (G1) $u$ is increasing and $u(0) = 1$, (G2) $\limsup_{r \to \infty} r^{-1} \log u(r) < \infty$, (G3) $\log u(x^2)$ is convex for $x \in [0, \infty)$. Such a function $u$ determines a Gel'fand triple $[\mathcal{E}]_u \subseteq (L^2) \subseteq [\mathcal{E}]_w^*$. In this paper we show that if $w$ is a positive continuous function on $[0, \infty)$ satisfying the conditions (G1) (G3) and (G*2) $\liminf_{r \to \infty} r^{-1} \log w(r) > 0$. Then its dual Legendre transform $u^* = w^*$ satisfies conditions (0) (G1) (G2) (G3). The function $w$ is often convenient to use for studying generalized functions on a white noise space.

1. INTRODUCTION

In recent papers [1] [2] [3] Asai, Kubo, and Kuo have introduced a Gel'fand triple $[\mathcal{E}]_u \subseteq (L^2) \subseteq [\mathcal{E}]_w^*$ basing on a Gel'fand triple $\mathcal{E} \subseteq \mathcal{E}_0 \subseteq \mathcal{E}'$ and a growth function $u$ satisfying certain conditions (see Section 2.) This function $u$ is used in [3] to characterize test functions in the space $[\mathcal{E}]_u$ and to give an intrinsic topology for $[\mathcal{E}]_w$. Moreover, its dual Legendre transform $u^*$ is used in the characterization theorem for generalized functions in the space $[\mathcal{E}]_w^*$.

It often happens, e.g., the Bell number spaces [1] [3] and Feynman integrals [4], that the function $u$ is given implicitly as the dual Legendre transform $w^*$ of some function $w$, i.e., $u = w^*$. Thus it is desirable to specify conditions on $w$ so that $u = w^*$ determines a Gel'fand triple $[\mathcal{E}]_u \subseteq (L^2) \subseteq [\mathcal{E}]_w^*$.

2. SPACES OF TEST AND GENERALIZED FUNCTIONS

Take a real countably-Hilbert space $\mathcal{E}$ with a sequence $\{ \cdot, \cdot \}_p$ of norms [6]. Let $\mathcal{E}_p$ denote the completion of $\mathcal{E}$ with respect to the norm $\| \cdot \|_p$. Assume the following conditions:

(a) There exists $0 < p < 1$ such that $\| \cdot\|_0 \leq \rho \| \cdot\|_1 \leq \cdots \leq \rho^p \| \cdot\|_p \leq \cdots$ for all $p \geq 0$.
(b) For any $p \geq 0$, there exists $q \geq p$ such that the inclusion map $i_{q,p} : \mathcal{E}_q \hookrightarrow \mathcal{E}_p$ is a Hilbert-Schmidt operator.

Let $\mathcal{E}'$ be the dual space of $\mathcal{E}$. Then we have a Gel'fand triple

$$\mathcal{E} \subseteq \mathcal{E}_0 \subseteq \mathcal{E}'$$

Apply the Minlos theorem to get the standard Gaussian measure $\mu$ on $\mathcal{E}'$. For simplicity, we use $(L^2)$ to denote the complex Hilbert space $L^2(\mathcal{E}', \mu)$. By the Wiener-Itô theorem, each $\varphi \in (L^2)$ can be uniquely decomposed into an orthogonal
sum as follows:

\[
\varphi = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} \langle \circ^n \cdot, f_n \rangle, \quad f_n \in C_{0,c}^\circ,
\]

where \( I_n \) is the multiple Wiener integral of order \( n \), \( \circ^n \cdot \) is the Wick tensor \([6]\), and the sub-index \( c \) denotes the complexification. In addition, the \( (L^2) \)-norm of \( \varphi \) is given by

\[
\|\varphi\|_0 = \left( \sum_{n=0}^{\infty} n! |f_n|^2_0 \right)^{1/2}.
\]

Let \( C_{+,1/2} \) be the set of positive continuous functions \( u \) on \([0, \infty)\) such that

\[
\lim_{r \to \infty} \frac{\log u(r)}{\sqrt{r}} = \infty.
\]

We need to consider the following conditions on a function \( u \in C_{+,1/2} \):

1. (G1) \( u \) is increasing and \( u(0) = 1 \).
2. (G2) \( \limsup_{r \to \infty} r^{n-1} \log u(r) < \infty \).
3. (G3) \( \log u(x^2) \) is convex for \( x \in [0, \infty) \).

These G-conditions are stated as U-conditions in the papers \([1]\) \([2]\) \([3]\). The only difference is between conditions (G2) and (U2). Condition (U2) says that \( \lim_{r \to \infty} r^{n-1} \log u(r) < \infty \). However, it can be replaced by the weaker condition (G2) in these papers.

Let \( u \in C_{+,1/2} \). Its \textit{Legendre transform} \( \ell_u \) and \textit{dual Legendre transform} \( u^* \) are defined as in \([1]\) by

\[
\ell_u(n) = \inf_{r > 0} \frac{u(r)}{r^n}, \quad n = 0, 1, 2, \ldots , \quad (2)
\]

\[
u^*(r) = \sup_{s > 0} \frac{e^{2\sqrt{rs}}}{u(s)}, \quad r \in [0, \infty) . \quad (3)
\]

We have the following facts about Legendre and dual Legendre transforms:

**Fact 1.** If \( u \in C_{+,1/2} \) satisfies (G1)-(G3), then \( \lim_{n \to \infty} \ell_u(n) 1/n = 0 \).

(See Lemma 3.6 in \([1]\) or Theorem 2.3 in \([3]\).)

By this fact, the following function \( L_u \) is an entire function

\[
L_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^n . \quad (4)
\]

**Fact 2.** If \( u \in C_{+,1/2} \) is increasing and satisfies (G3), then \( u \) and \( L_u \) are equivalent, namely, there exist constants \( K_1, K_2, a_1, a_2 > 0 \) such that

\[
K_1 u(a_1 r) \leq L_u(r) \leq K_2 u(a_2 r), \quad \forall r \in [0, \infty) . \quad (5)
\]

(See Theorem 3.13 in \([1]\) or Theorem 2.6 in \([3]\).)

**Fact 3.** If \( u \in C_{+,1/2} \) satisfies (G3), then \( \lim_{n \to \infty} (\ell_u(n)(n!)^2)^{-1/n} = 0 \).

(See Lemma 3.2 in \([3]\).)

By this fact, the following function \( L^\#_u \) is an entire function

\[
L^\#_u(r) = \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)(n!)^2} r^n . \quad (6)
\]
Fact 4. If \( u \in C_{+,1/2} \), then \( u^* \) belongs to \( C_{+,1/2} \) and is increasing and satisfies (G3). If in addition \( u \) satisfies (G1), then \( u^* \) also satisfies (G1).
(See Lemma 4.5 in [1] or Theorem 2.7 in [3].)

Fact 5. If \( u \in C_{+,1/2} \) satisfies (G1) (G3), then \( (u^*)^* = u \) on \([0, \infty)\).
(See Theorem 4.7 in [1] or Theorem 2.9 in [3].)

Fact 6. If \( u \in C_{+,1/2} \), then \( u^* \) and \( \mathcal{L}_u^* \) are equivalent. If in addition \( u \) satisfies (G3), then \( \mathcal{L}_u^* \) and \( \mathcal{L}_u^\# \) are equivalent.
(See above Facts 2 and 4, and Theorem 4,10 in [1].)

Now, let \( u \in C_{+,1/2} \) be a fixed function satisfying conditions (G1) (G2) (G3). For each \( p \geq 0 \) and for \( \varphi \) given by Equation (1), define \( \| \varphi \|_{p,u} \) by

\[
\| \varphi \|_{p,u} = \left( \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2 \right)^{1/2},
\]

where \( \ell_u \) is the Legendre transform of \( u \) defined by Equation (2).

Let \( [\mathcal{E}]_u = \{ \varphi \in (L^2) \mid \| \varphi \|_{p,u} < \infty \} \) and let \( [\mathcal{E}]_u^\ast \) be the projective limit of \( \{ [\mathcal{E}]_u \mid p \geq 0 \} \). This space \( [\mathcal{E}]_u \) is the space of test functions on \( \mathcal{E} \) given by the growth function \( u \). Its dual space \( [\mathcal{E}]_u^\ast \) is the space of generalized functions on \( \mathcal{E} \).

By using conditions (a) and (G2) we can show that \( [\mathcal{E}]_u \subset (L^2) \) for all large \( p \) (see Section 3 in [3] for the proof.) Hence \( [\mathcal{E}]_u \subset (L^2) \). Moreover, Condition (b) implies that \( [\mathcal{E}]_u \) is a nuclear space. The space \( (L^2) \) can be identified with its dual space and so we get a Gel’fand triple

\[
[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^\ast.
\]

Note that this Gel’fand triple is exactly the CKS-space \([\mathcal{E}]_\alpha \subset (L^2) \subset [\mathcal{E}]_\alpha^\ast\) in [5] given by the sequence

\[
\alpha_u(n) = \frac{1}{n!\ell_u(n)}, \quad n = 0, 1, 2, \ldots
\]

Let \( \xi \in \mathcal{E}_\alpha \). The renormalized exponential function \( e^{(\cdot, \xi)} : \) is defined by

\[
e^{(\cdot, \xi)} := \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\xi^\otimes_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle (\cdot, (\xi^\otimes_n) \rangle.
\]

Use Equations (6) and (7) to get

\[
\| e^{(\cdot, \xi)} \|_{p,u} = \left( \sum_{n=0}^{\infty} \frac{\ell_u(n)}{(\ell_u(n))^2} |\xi|_p^2 \right)^{1/2} = \mathcal{L}_u^\# (|\xi|_p^2)^{1/2}.
\]

Hence \( \| e^{(\cdot, \xi)} \|_{p,u} < \infty \) for all \( p \geq 0 \) and so \( e^{(\cdot, \xi)} \in [\mathcal{E}]_u \) for any \( \xi \in \mathcal{E}_\alpha \).

The \( S \)-transform of a generalized function \( \Phi \) in \([\mathcal{E}]_\alpha^\ast\) is the function defined by

\[
S\Phi(\xi) = \langle \Phi, e^{(\cdot, \xi)} \rangle, \quad \xi \in \mathcal{E}_\alpha,
\]

where \( \langle \cdot, \cdot \rangle \) is the bilinear pairing of \([\mathcal{E}]_u^\ast \) and \([\mathcal{E}]_u \).

Let \( \Phi \in [\mathcal{E}]_\alpha^\ast \). By the continuity of \( \Phi \) there exist constants \( K, p > 0 \) such that

\[
\| \langle \Phi, \varphi \rangle \|_{p,u} \leq K \| \varphi \|_{p,u}, \quad \forall \varphi \in [\mathcal{E}]_u.
\]

Put \( \varphi = e^{(\cdot, \xi)} \) and use Equation (9) to get

\[
|S\Phi(\xi)| \leq K \mathcal{L}_u^\# (|\xi|_p^2)^{1/2}, \quad \forall \xi \in \mathcal{E}_\alpha.
\]
But from the above Fact 6 the function $L_u^p$ is equivalent to $u^*$. Hence the inequality in Equation (10) is equivalent to the existence of constants $K, a, p > 0$ such that
\[ |S\Phi(\xi)| \leq K u^*(a|\xi|^2_p)^{1/2}, \quad \forall \xi \in \mathcal{E}_c. \]
This is the growth condition in the following theorem due to Asai-Kubo-Kuo (see Theorem 3.4 in [3]).

**Theorem 2.1.** Assume that $u \in C_{+,1/2}$ satisfies (G1) (G2) (G3). Then a function $F: \mathcal{E}_c \to \mathbb{C}$ is the $S$-transform of a generalized function in $[\mathcal{E}]_u$ if and only if it satisfies the conditions:

(a) For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.

(b) There exist constants $K, a, p \geq 0$ such that
\[ |F(\xi)| \leq K u^*(a|\xi|^2_p)^{1/2}, \quad \forall \xi \in \mathcal{E}_c. \]

Next consider test functions. Let $\varphi \in [\mathcal{E}]_u$ be represented as in Equation (1). Then its $S$-transform is given by
\[
S\varphi(\xi) = \left\langle \sum_{n=0}^{\infty} \langle \xi|^{\otimes n}, f_n \rangle, \sum_{n=0}^{\infty} \frac{1}{n!} \langle \zeta|^{\otimes n}, \xi^{\otimes n} \rangle \right\rangle \\
= \sum_{n=0}^{\infty} \frac{1}{n!} \langle f_n, \varphi(\xi) \rangle \\
= \sum_{n=0}^{\infty} \langle f_n, \xi^{\otimes n} \rangle, \quad \xi \in \mathcal{E}_c.
\]

Therefore, for any $p \geq 0$,
\[
|S\varphi(\xi)| \leq \sum_{n=0}^{\infty} |f_n|_p |\xi|_p^n \\
= \sum_{n=0}^{\infty} \left( \frac{1}{\ell_u(n)} |f_n|_p \right) \left( \sqrt{\ell_u(n)} |\xi|_p^n \right).
\]

Apply the Schwarz inequality and use Equations (4) and (7) to get
\[
|S\varphi(\xi)| \leq \left( \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^n \right)^{1/2} \left( \sum_{n=0}^{\infty} \ell_u(n) |\xi|_p^n \right)^{1/2} \\
= \|\varphi\|_{p,u} L_u(|\xi|^2_p)^{1/2}. \tag{11}
\]

But by the above Fact 2 the function $L_u$ is equivalent to $u^*$. Hence the inequality in Equation (11) is equivalent to the statement: For any constants $a, p \geq 0$, there exists a constant $K \geq 0$ such that
\[ |S\Phi(\xi)| \leq K u^*(a|\xi|^2_p)^{1/2}, \quad \forall \xi \in \mathcal{E}_c. \]

This is the growth condition in the next theorem due to Asai-Kubo-Kuo from Theorem 3.6 in [3].

**Theorem 2.2.** Assume that $u \in C_{+,1/2}$ satisfies (G1) (G2) (G3). Then a function $F: \mathcal{E}_c \to \mathbb{C}$ is the $S$-transform of a test function in $[\mathcal{E}]_u$ if and only if it satisfies the conditions:
(a) For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.
(b) For any constants $a, p \geq 0$, there exists a constant $K \geq 0$ such that
\[
|F(\xi)| \leq Ku\left(a|\xi|^p\right)^{1/2}, \quad \forall \xi \in \mathcal{E}_c.
\]

We give a well-known example for Theorems 2.1 and 2.2. Let $0 \leq \beta < 1$ and consider the function
\[
u(r) = \exp \left( (1 + \beta)r \right), \quad r \in [0, \infty).
\]
It is easy to check that $\nu$ belongs to $C_{+1/2}$ and satisfies conditions (G1) (G2) (G3). Moreover, the Legendre and dual Legendre transforms of $\nu$ are given by
\[
\ell_{\nu}(n) = \left( \frac{\Gamma(n+1)}{n} \right)^{(1+\beta)}, \quad n = 0, 1, 2, \ldots , \quad (12)
\]
\[
u^*(r) = \exp \left( (1 - \beta)r \right), \quad r \in [0, \infty), \quad (13)
\]
where $0^0 = 1$ by convention.

We can use the Stirling formula to show that the sequence $\alpha_{\nu}(n) = (n\ell_{\nu}(n))^{-1}$ given by Equations (8) and (12) is equivalent to the sequence $\alpha(n) = (n!)^\beta$. Thus the Gel'fand triple $[\mathcal{E}] \subset (L^2) \subset [\mathcal{E}]^\ast_\beta$ is exactly the triple $(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})^\ast_\beta$ introduced by Kondratiev-Streit (see [6]). In this case Theorems 2.1 and 2.2 are due to Kondratiev-Streit (see Theorems 8.2 and 8.10 in [6]).

Note that when $\beta = 1$, the function $\nu(r) = e^{2\sqrt{r}}$ does not belong to $C_{+1/2}$ and so its dual Legendre transform $\nu^*$ is not defined. This fact is also evident from Equation (13).

3. GROWTH FUNCTIONS FOR GENERALIZED FUNCTIONS

Consider the Bell number spaces introduced in [5]. Let $\exp_k(r)$ be the $k$-th iterated exponential function, $k \geq 1$. It has the power series expansion
\[
\exp_k(r) = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} r^n.
\]
Let $b_k(n) = B_k(n)/\exp_k(0)$. The numbers in the sequence $\{b_k(n)\}_{n=0}^{\infty}$ are called Bell numbers of order $k$.

Put $w_k(r) = \exp_k(r)/\exp_k(0)$. Then we have
\[
w_k(r) = \sum_{n=0}^{\infty} \frac{b_k(n)}{n!} r^n. \quad (14)
\]

The Bell number space is the CKS-space $[\mathcal{E}]_{k, B} \subset (L^2) \subset [\mathcal{E}]^\ast_{k, B}$ given by the sequence $\{b_k(n)\}_{n=0}^{\infty}$. To view the Bell number space as a Gel'fand triple in Section 2, we need to find a growth function $u \in C_{+1/2}$ satisfying (G1) (G2) (G3). In order to find such a function $u$, note that the growth condition (b) in the characterization theorem in [5] for generalized functions in $[\mathcal{E}]^\ast_\beta$ takes the form: There exist constants $K, a, p \geq 0$ such that
\[
|F(\xi)| \leq Kw_k(a|\xi|^p)^{1/2}, \quad \forall \xi \in \mathcal{E}_c.
\]
By comparing this growth condition with the one in Theorems 2.1, we see that we may take $u$ such that $u^* = w_k$. If we can check that $u$ (assuming its existence) belongs to $C_{+1/2}$ and satisfies (G1) (G2) (G3), then by the above Fact 5 we get
u = (u^*)^* = u^*_w. Thus the function u is given by u = w^*_w, the dual Legendre transform of w_w. However, it is impossible to find the explicit form of w^*_w.

From the above discussion we see that it is desirable to find conditions on u^* (instead of u) so that [E]_u \subset (L^2) \subset [E]_u^* is a Gel'fand triple as given in Section 2. For this purpose we will need to consider the condition:

\[(G^*2) \lim \inf_{r \to \infty} r^{-1} \log w(r) > 0.\]

The following three lemmas can be easily checked.

**Lemma 3.1.** Let u \in C_{+,1/2}. Then u satisfies condition (G2) if and only if there exist constants c_1, c_2 > 0 such that u(r) \leq c_1 e^{c_2 r} for all r \geq 0.

**Lemma 3.2.** Let w be a positive continuous function on [0, \infty). If there exist constants c_1, c_2 > 0 such that w(r) \geq c_1 e^{c_2 r} for all r \geq 0, then w \in C_{+,1/2}.

**Lemma 3.3.** Let w be a positive continuous function on [0, \infty). Then w satisfies (G*2) if and only if there exist constants c_1, c_2 > 0 such that w(r) \geq c_1 e^{c_2 r} for all r \geq 0.

Now, we state and prove the main theorem in this paper.

**Theorem 3.4.** If u \in C_{+,1/2} satisfies (G1) (G2) (G3), then u^* satisfies (G1) (G^*2) (G3). Conversely, if w is a positive continuous function on [0, \infty) satisfying (G1) (G^*2) (G3), then w^* belongs to C_{+,1/2} and satisfies (G1) (G2) (G3).

**Remark.** Note that if w is a positive continuous function on [0, \infty) satisfying (G^*2), then u belongs to C_{+,1/2} by Lemmas 3.2 and 3.3. Hence the dual Legendre transform u^* is defined.

**Proof.** Assume that u \in C_{+,1/2} satisfies (G1) (G2) (G3). We can use Fact 4 in Section 2 to see that u^* satisfies (G1) (G3). On the other hand, by Lemma 3.1 there exist constants c_1, c_2 > 0 such that

\[u(r) \leq c_1 e^{c_2 r}, \quad \forall r \geq 0.\]

Therefore, by Equation (3),

\[u^*(r) = \sup_{s>0} \frac{e^{2\sqrt{rs}}}{u(s)} \geq \frac{1}{c_1} \sup_{s>0} c_2 (2\sqrt{rs} - c_2 s), \quad \forall r \geq 0.\]

But it is easy to check that \sup_{s>0} (2\sqrt{rs} - c_2 s) = r/c_2. Thus we get

\[u^*(r) \geq \frac{1}{c_1} e^{r/c_2}, \quad \forall r \geq 0.\]

Hence by Lemma 3.3 u^* satisfies (G^*2).

Conversely, assume that w is a positive continuous function on [0, \infty) satisfying (G1) (G^*2) (G3). By Lemmas 3.2 and 3.3 the function w belongs to C_{+,1/2}. Since w satisfies (G1) by assumption, we can use Fact 4 in Section 2 to see that w^* satisfies (G1) (G3). Moreover, by Lemma 3.3 there exist constants c_1, c_2 > 0 such that

\[w(r) \geq c_1 e^{c_2 r}, \quad \forall r \geq 0.\]
Then by Equation (3)
\[ w^*(r) = \sup_{s>0} \frac{e^{2\sqrt{s}}}{w(s)} \leq \frac{1}{c_1} \sup_{s>0} e^{2\sqrt{s} - c_2 s} = \frac{1}{c_1} e^{r/c_2}, \quad \forall r \geq 0. \]

Hence by Lemma 3.1 the function \( w^* \) satisfies (G2).

As a simple example, consider the function \( w_k(r) = \exp_k(r)/\exp_k(0) \) given in Equation (14) for the Bell number spaces. Obviously, \( w_k \) is a positive continuous function on \([0, \infty)\). Moreover, it is easy to check that \( w_k \) satisfies (G1) (G*2) (G3). Thus by Theorem 3.4 \( w_k^* \) belongs to \( C_{+1/2} \) and satisfies (G1) (G2) (G3). Hence the function \( u_k = w_k^* \) determines a Gel'fand triple \([\mathcal{E}]_{u_k} \subset (L^2) \subset [\mathcal{E}]_u^* \). This Gel'fand triple turns out to be exactly the Bell number space associated with the sequence \( \{b_k(n)\}_{n=0}^\infty \) (See Example 4.3 in [7].)

An interesting example of \( w \) is given in a recent paper by Asai-Kubo-Kuo [4] on Feynman integrals. Let \( \nu \) be a complex measure on \( \mathbb{R} \) with total variation \( |\nu| \).

Assume that \( \nu \) satisfies the conditions:

1. \( |\nu|(\mathbb{R} \setminus \{0\}) > 0 \).
2. \( \int_{\mathbb{R}} e^{c|\lambda|} d|\nu|(|\lambda|) < \infty \) for any constant \( c > 0 \).

By condition (2) we can define a function \( w \) by
\[ w(r) = \exp \left( \int_{\mathbb{R}} (e^{r|\lambda|} - 1) d|\nu|(|\lambda|) \right), \quad r \in [0, \infty). \]  

(15)

Obviously, \( w \) is a positive continuous function on \([0, \infty)\). It is easy to see that \( w \) satisfies (G1) (G3). To check condition (G*2), note that \( e^x - 1 \geq \frac{1}{2} x^2 \) for \( x \geq 0 \) and so
\[ \log w(r) \geq \frac{1}{2} r \int_{\mathbb{R}} \lambda^2 d|\nu|(|\lambda|), \quad \forall r \geq 0. \]

Hence we have the inequality
\[ \liminf_{r \to \infty} \frac{\log w(r)}{r} \geq \frac{1}{2} \int_{\mathbb{R}} \lambda^2 d|\nu|(|\lambda|). \]

(16)

It follows from condition (1) that \( \int_{\mathbb{R}} \lambda^2 d|\nu|(|\lambda|) > 0 \). Hence the function \( w(r) \) satisfies condition (G*2).

Thus the function \( w(r) \) defined in Equation (15) satisfies (G1) (G*2) (G3). Then by Theorem 3.4 the function \( u = w^* \) belongs to \( C_{+1/2} \) and satisfies (G1) (G2) (G3).

With this function \( u \) we have a Gel'fand triple
\[ [\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*. \]

For such a complex measure \( \nu \) on \( \mathbb{R} \), it has been shown in [4] that the Feynman integrands associated with the following potentials
\[ V(x) = \int_{\mathbb{R}} e^{i\lambda x} d\nu(|\lambda|), \quad V(x) = \int_{\mathbb{R}} e^{\lambda x} d\nu(|\lambda|) \]
are generalized functions in the space \([\mathcal{S}]_u^* \), given by \( u = w^* \) with \( w \) defined by Equation (15). Here \( \mathcal{S} \) is the Schwartz space replacing the countably-Hilbert space \( \mathcal{E} \) in Section 1.
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