Diagonalization of the Lévy Laplacian
and Related Stable Processes

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Abstract

In this paper, we introduce diagonalization of the Lévy Laplacian along its eigenfunctions. We describe new Hilbert spaces as various domains of the Lévy Laplacian and construct the corresponding equi-continuous semigroups of class \((C_0)\). Moreover, we discuss infinite dimensional stochastic processes related to these extensions and one-dimensional stable processes.

1. Introduction

An infinite dimensional Laplacian, the Lévy Laplacian, was introduced by P. Lévy [17]. This Laplacian was discussed in the framework of white noise analysis by T. Hida [4]. It has been studied by many authors, see [1, 2, 3, 5, 7, 8, 13, 15, 16, 18, 21, 22, 23, 24], among others. In particular, L. Accardi et al. [1] obtained an important relationship between this Laplacian and the Yang-Mills equations.

In the previous papers [27, 28], we introduced a Hilbert space as a domain of the Lévy Laplacian and extended the Laplacian to a self-adjoint operator. We obtained stochastic processes generated by the powers of an extended Lévy Laplacian in [25, 26] and by some functions of the Laplacian in [29].

In this paper we will introduce new Hilbert spaces consisting of eigenfunctions of the Lévy Laplacian acting on generalized white noise functionals and
discuss the self-adjointness of the Laplacian on each Hilbert space. These Hilbert spaces are generalizations of those previous Hilbert spaces in [25]-[28]. We will show that each extension of the Lévy Laplacian generates a different stochastic process. Thus the stochastic process generated by the Lévy Laplacian depends on the choice of eigenfunctions of the Laplacian.

The paper is organized as follows. In Section 2 we give a brief background in white noise analysis which is necessary for our paper. In Section 3 we introduce Hilbert spaces $E^\gamma_{p,N}$, $p \geq 1, N \geq 1, 1 \leq \gamma \leq 2$, as new domains of the Lévy Laplacian, and for each $1 \leq \gamma \leq 2$, we give an equi-continuous semigroup $\{G^\gamma_t; t \geq 0\}$ of class $(C_0)$ generated by the Laplacian defined on $E^\gamma_{p,\infty} = \bigcap_{N \geq 1} E^\gamma_{p,N}$ with the projective limit topology. In Section 4, using a strictly stable process $\{X^\gamma_t; t \geq 0\}, 1 \leq \gamma \leq 2$ with each characteristic function given by $E[e^{i\xi X^\gamma_t}] = e^{-t|\xi|^\gamma}$, we give an infinite dimensional stochastic process $\{X^\gamma_t; t \geq 0\}$ generated by the Laplacian on $E^\gamma_{p,\infty}$, which satisfies $\tilde{G}^\gamma_t F(\xi) = E[F(X^\gamma_t)|X^\gamma_0 = \xi]$ for all $F \in \mathcal{S}(E^\gamma_{p,\infty})$, where $\mathcal{S}$ is the $S$-transform and $\tilde{G}^\gamma_t = SG^\gamma_t S^{-1}$. Moreover we will give an operator-valued stochastic process associated with the Lévy Laplacian acting on $E^\gamma_{p,\infty}$ for each $1 \leq \gamma \leq 2$. It is important to observe that the stochastic process generated by the Lévy Laplacian depends on the choice of eigenfunctions of the Laplacian. Changing the domain consisting of eigenfunctions of the Lévy Laplacian will produce a different stochastic process generated by the Laplacian.

2. Preliminaries

In this section we assemble some basic notations of white noise analysis following [7, 12, 15, 19].

We take the space $E^* \equiv \mathcal{S}'(\mathbb{R})$ of tempered distributions with the standard Gaussian measure $\mu$ which satisfies

$$\int_{\mathbb{R}} \exp \{ i \langle x, \xi \rangle \} \, d\mu(x) = \exp \left( -\frac{1}{2} |\xi|^2 \right), \quad \xi \in E \equiv \mathcal{S}(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $E^* \times E$ and $| \cdot |_0$ is the $L^2(\mathbb{R})$-norm.

Let $A = -(d/dx)^2 + u^2 + 1$. This is a densely defined self-adjoint operator on $L^2(\mathbb{R})$ and there exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for $L^2(\mathbb{R})$ such that $Ae_\nu = 2(\nu + 1)e_\nu$. We define the norm $| \cdot |_p$ by $|f|_p = |A^p f|_0$ for $f \in E$ and $p \in \mathbb{R}$, and let $E_p$ be the completion of $E$ with respect to the norm $| \cdot |_p$. Then $E_p$ is a real separable Hilbert space with the norm $| \cdot |_p$ and the dual space $E'_p$ of $E_p$ is the same as $E_{-p}$ (see [10]). The space $E$ is the projective limit space of $\{E^*_p; p \geq 0\}$ and $E^*$ is the inductive limit space of $\{E_{-p}; p \geq 0\}$, and $E$ becomes a nuclear space with the Gel'fand triple $E \subset L^2(\mathbb{R}) \subset E^*$. We denote the complexifications of $L^2(\mathbb{R})$, $E$ and $E_p$ by $L^2_{\mathbb{C}}(\mathbb{R})$, $E_{\mathbb{C}}$ and $E^*_p, E_p, \mathbb{C}$, respectively.

The space $(L^2) = L^2(E^*, \mu)$ of complex-valued square-integrable functionals
defined on $E^*$ admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where $H_n$ is the space of multiple Wiener integrals of order $n \in \mathbb{N}$ and $H_0 = \mathbb{C}$. Let $L^2_{\mathbb{C}}(\mathbb{R})^{\otimes^n}$ denote the $n$-fold symmetric tensor product of $L^2_{\mathbb{C}}(\mathbb{R})$. If $\varphi \in (L^2)$ has the representation $\varphi = \sum_{n=0}^{\infty} \mathbf{1}_n(f_n)$, $f_n \in L^2_{\mathbb{C}}(\mathbb{R})^{\otimes^n}$, then the $(L^2)$-norm $||\varphi||_0$ is given by

$$||\varphi||_0 = \left(\sum_{n=0}^{\infty} n! |f_n|^2 \right)^{1/2},$$

where $| \cdot |_0$ is the $L^2_{\mathbb{C}}(\mathbb{R})^{\otimes^n}$-norm.

For $p \in \mathbb{R}$, let $||\varphi||_p = ||\Gamma(A)^p \varphi||_0$, where $\Gamma(A)$ is the second quantization operator of $A$. If $p \geq 0$, let $(E)_p$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(E)_p$ be the completion of $(L^2)$ with respect to the norm $|| \cdot ||_p$. Then $(E)_p$, $p \in \mathbb{R}$, is a Hilbert space with the norm $|| \cdot ||_p$. It is easy to see that for $p > 0$, the dual space $(E)^*_p$ of $(E)_p$ is given by $(E)_{-p}$. Moreover, for any $p \in \mathbb{R}$, we have the decomposition

$$(E)_p = \bigoplus_{n=0}^{\infty} H^{(p)}_n,$$

where $H^{(p)}_n$ is the completion of $\{ \mathbf{1}_n(f); f \in E_{\mathbb{C}}^{\otimes^n} \}$ with respect to $|| \cdot ||_p$. Here $E_{\mathbb{C}}^{\otimes^n}$ is the $n$-fold symmetric tensor product of $E_{\mathbb{C}}$. We also have $H^{(p)}_n = \{ \mathbf{1}_n(f); f \in E_{\mathbb{C}}^{\otimes^n} \}$ for any $p \in \mathbb{R}$, where $E_{\mathbb{C}}^{\otimes^n}$ is also the $n$-fold symmetric tensor product of $E_{\mathbb{C}}$. The norm $||\varphi||_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{1}_n(f_n) \in (E)_p$ is given by

$$||\varphi||_p = \left(\sum_{n=0}^{\infty} n! |f_n|^2 \right)^{1/2}, \quad f_n \in E_{\mathbb{C}}^{\otimes^n},$$

where the norm on $E_{\mathbb{C}}^{\otimes^n}$ is denoted also by $| \cdot |_p$.

The projective limit space $(E)$ of spaces $(E)_p$, $p \in \mathbb{R}$ is a nuclear space. The inductive limit space $(E)^*$ of spaces $(E)^*_p$, $p \in \mathbb{R}$ is nothing but the strong dual space of $(E)$. The space $(E)^*$ is called the space of generalized white noise functionals. We denote by $\langle \cdot , \cdot \rangle$ the canonical bilinear form on $(E)^* \times (E)$. Then we have

$$\langle \Phi , \varphi \rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{1}_n(F_n) \in (E)^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{1}_n(f_n) \in (E)$, where the canonical bilinear form on $(E_{\mathbb{C}}^{\otimes^n})^* \times (E_{\mathbb{C}}^{\otimes^n})$ is denoted also by $\langle \cdot , \cdot \rangle$.

Since $\exp(\cdot , \xi) \in (E)$, we can define the $S$-transform on $(E)^*$ by

$$S[\Phi](\xi) = \exp \left( -\frac{1}{2} \langle \xi , \xi \rangle \right) \langle \Phi , \exp(\cdot , \xi) \rangle, \quad \xi \in E_{\mathbb{C}}.$$
3. An equi-continuous semigroup of class \( (C_0) \) generated by the Lévy Laplacian

Let \( \Phi \) be in \( (E)^* \). Then the \( S \)-transform \( S[\Phi] \) of \( \Phi \) is Fréchet differentiable, i.e.

\[
S[\Phi](\xi + \eta) = S[\Phi](\xi) + S[\Phi]'(\xi)(\eta) + o(\eta),
\]

where \( o(\eta) \) means that there exists \( p \geq 0 \) depending on \( \xi \) such that \( o(\eta)/|\eta|^p \to 0 \)
as \( |\eta| \to 0 \).

We fix a finite interval \( T \) in \( \mathbb{R} \). Take an orthonormal basis \( \{\zeta_n\}_{n=0}^\infty \subset E \) for \( L^2(T) \) satisfying the equal density and uniform boundedness property (see e.g., [7, 15, 16, 18, 24]). Let \( D_L \) denote the set of all \( \Phi \in (E)^* \) such that the limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi](\zeta_n, \zeta_n)
\]
exists for any \( \xi \in E^* \) and is in \( S[(E)^*] \). The Lévy Laplacian \( \Delta_L \) is defined by

\[
\Delta_L \Phi = S^{-1} \tilde{\Delta}_L S \Phi
\]
for \( \Phi \in D_L \). We denote by \( D_L^T \) the set of all functionals \( \Phi \in D_L \) such that
\( S[\Phi](\eta) = 0 \) for all \( \eta \in E \) with \( \text{supp}(\eta) \subset T^c \).

Take a generalized white noise functional

\[
\Phi(x) = \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) : e^{ia_1 z(u_1)} \ldots e^{ia_n z(u_n)} : \, du,
\]

where \( : \) denotes the Wick ordering. If \( \Phi \) as in (3.1) belongs to \( D_L^T \), then \( \Phi \) is equal to

\[
\int_{T^n} f(u_1, \ldots, u_n) : e^{ia_1 z(u_1)} \ldots e^{ia_n z(u_n)} : \, du.
\]

Hence the \( S \)-transform \( S[\Phi] \) of \( \Phi \) is given by

\[
S[\Phi](\xi) = \int_{T^n} f(u)e^{ia_1 \xi(u_1)} \ldots e^{ia_n \xi(u_n)} \, du.
\]

This functional is important as an eigenfunction of the operator \( \Delta_L \). In fact, we have the following result.

**Theorem 3.1.** [27] A generalized white noise functional \( \Phi \) as in (3.1) satisfies the equation

\[
\Delta_L \Phi = -\frac{1}{|T|} \left( \sum_{\nu=1}^n a_{\nu} \right) \Phi.
\]

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For each $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$, and $1 \leq \gamma \leq 2$, let

$$A_{\lambda,n}^\gamma = \left\{ (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n; \sum_{\nu=1}^n a_\nu = \lambda, \sum_{\nu=1}^n a_\nu^2 = |\lambda|^2 \right\},$$

$$D_{\lambda,n}^\gamma = \text{LS} \left\{ \int_{T^n} f(u) : \prod_{\nu=1}^n e^{i a_\nu x(u_\nu)} : du \in D_{L}^2; \right\},$$

$$f \in E_{C}^{\otimes n}, (a_1, a_2, \ldots, a_n) \in A_{\lambda,n}^\gamma,$$

where LS means the linear span.

From now on, $\gamma \in [1, 2]$ will be an arbitrarily fixed number. Set $D_{\lambda,0}^\gamma = \mathbb{C}$ and let

$$D_{\lambda}^\gamma = \text{LS} \left\{ D_{\lambda,n}^\gamma; \ n \in \mathbb{N} \right\}.$$

Then $D_{\lambda}^\gamma$ is a linear subspace of $(E)_{-p}$ for any $p \geq 1$, and $\Delta_L$ is a linear operator from $D_{\lambda}^\gamma$ into itself such that $\Delta_L \Phi = -\frac{|\lambda|^2}{|T|^2} \Phi$ for any $\Phi \in D_{\lambda}^\gamma$. For convenience, we will use the following notation

$$J_n[f](x) = \int_{T^n} f(u) : \prod_{\nu=1}^n e^{i a_\nu x(u_\nu)} : du.$$

Define a space $\overline{D_{\lambda}^\gamma}$ by the completion of $D_{\lambda}^\gamma$ in $(E)_{-p}$ with respect to $|| \cdot ||_{-p}$. Then for each $n \in \mathbb{N} \cup \{0\}$, $\overline{D_{\lambda}^\gamma}$ becomes a Hilbert space with the inner product of $(E)_{-p}$ and the Lévy Laplacian $\Delta_L$ becomes a continuous linear operator $\Delta_L$ from $\overline{D_{\lambda}^\gamma}$ into itself satisfying

$$\Delta_L \Phi = -\frac{|\lambda|^2}{|T|^2} \Phi$$

for any $\Phi \in \overline{D_{\lambda}^\gamma}$.

The Lévy Laplacian $\Delta_L$ is a self-adjoint operator on $\overline{D_{\lambda}^\gamma}$ for each $\lambda \in \mathbb{R}$.

**Proposition 3.2.** (cf. [27]) Let $\Phi = \int_{\mathbb{R}} \Phi_\lambda d\lambda$ and $\Psi = \int_{\mathbb{R}} \Psi_\lambda d\lambda$ be generalized white noise functionals such that $\Phi_\lambda$ and $\Psi_\lambda$ are in $\overline{D_{\lambda}^\gamma}$ for each $\lambda \in \mathbb{R}$ and strongly measurable in $\lambda$. If $\Phi = \Psi$ in $(E)^*$, then $\Phi_\lambda = \Psi_\lambda$ in $(E)^*$ for almost all $\lambda \in \mathbb{R}$.

**Proof.** Let $A_{\lambda}^\gamma = \bigcup_n A_{\lambda,n}^\gamma$. Then, for almost all $\lambda \in \mathbb{R}$, $\Phi_\lambda$ and $\Psi_\lambda$ can be expressed in the forms:

$$\Phi_\lambda = \lim_{N \to \infty} \sum_{a^{(n)} \in A_{\lambda}^\gamma} J_{a^{(n)}}[f_{a^{(n)}}], \quad \Psi_\lambda = \lim_{N \to \infty} \sum_{a^{(n)} \in A_{\lambda}^\gamma} J_{a^{(n)}}[g_{a^{(n)}}],$$

where $\sum_{a^{(n)} \in A_{\lambda}^\gamma}$ means a sum of finitely many terms on $a^{(n)} \in A_{\lambda}^\gamma$. Suppose $\Phi = \Psi$ in $(E)^*$. Then, taking the $S$-transform, we have

$$\int_{\mathbb{R}} \lim_{N \to \infty} \sum_{a^{(n)} \in A_{\lambda}^\gamma} S(J_{a^{(n)}}[f_{a^{(n)}} - g_{a^{(n)}}])(\xi) d\lambda = 0$$

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for all \( x \in E_C \). Take \( \xi_T \in E_C \) such that \( \xi_T = 1 \) on \( T \) and put \( \xi = a \xi_T + \eta \) with \( a \in \mathbb{C} \) and \( \eta \in E_C \). Then we get

\[
\int_{\mathbb{R}} e^{i \lambda a} \lim_{N \to \infty} \sum_{a^{(N)} \in A^T} S \left( J_{a^{(N)}} [ f_{a^{(N)}} - g_{a^{(N)}} ] \right)(\eta) \, d\lambda = 0
\]

for all \( a \in \mathbb{C} \) and \( \eta \in E_C \). Therefore,

\[
\lim_{N \to \infty} \sum_{a^{(N)} \in A^T} S \left( J_{a^{(N)}} [ f_{a^{(N)}} - g_{a^{(N)}} ] \right)(\eta) = 0
\]

for all \( \lambda \in \mathbb{R} \) and \( \eta \in E_C \). This implies that \( \Phi_{\lambda} = \Psi_{\lambda} \) in \((E)^*\) for almost all \( \lambda \in \mathbb{R} \). \( \square \)

Proposition 3.2 says that \( \int_{\mathbb{R}} \Phi_{\lambda} \, d\lambda \) with \( \Phi_{\lambda} \in \mathcal{D}^\gamma_\lambda \) is uniquely determined as an element of \((E)^*\). Therefore, for any \( N \in \mathbb{N}, p \geq 1 \), we can define a space \( E^\gamma_{-p, N} \) by

\[
E^\gamma_{-p, N} = \left\{ \int_{\mathbb{R}} \Phi_{\lambda} \, d\lambda \in (E)^* : \int_{\mathbb{R}} \| \Phi_{\lambda} \|_{-p}^2 \alpha_N^\gamma(\lambda) \, d\lambda < \infty, \Phi_{\lambda} \in \mathcal{D}^\gamma_\lambda \forall \lambda \in \mathbb{R} \right\},
\]

where \( \alpha_N^\gamma(\lambda) = \sum_{\ell=0}^N \left( \frac{N!}{\ell !} \right)^{2\ell} \). Define a norm \( \| \cdot \|_{-p, N} \) on \( E^\gamma_{-p, N} \) by

\[
\| \Phi \|_{-p, N} = \left( \int_{\mathbb{R}} \| \Phi_{\lambda} \|_{-p}^2 \alpha_N^\gamma(\lambda) \, d\lambda \right)^{1/2}, \quad \Phi = \int_{\mathbb{R}} \Phi_{\lambda} \, d\lambda \in E^\gamma_{-p, N}.
\]

Then the space \( E^\gamma_{-p, N} \) is a Hilbert space with the norm \( \| \cdot \|_{-p, N} \) for each \( N \in \mathbb{N} \) and \( p \geq 1 \). We note that \( E^\gamma_{-p, N} \) is isomorphic to the direct integral \( \int d\lambda \mathcal{D}^\gamma_\lambda \).

Put \( E^\gamma_{-p, \infty} = \bigcap_{N \geq 1} E^\gamma_{-p, N} \), with the projective limit topology. Then, for any \( N \geq 1 \), we have the following inclusion relations:

\[
E^\gamma_{-p, \infty} \subset E^\gamma_{-p, N+1} \subset E^\gamma_{-p, N} \subset E^\gamma_{-p, 1} \subset (E)^{-p}.
\]

The Laplacian \( \Delta_L \) can be defined on \( E^\gamma_{-p, \infty} \) and is a continuous linear operator from \( E^\gamma_{-p, 2} \) into \( E^\gamma_{-p, 1} \) satisfying \( \| \Delta_L \Phi \|_{-p, N} \leq \| \Phi \|_{-p, N+1} \) for all \( \Phi \in E^\gamma_{-p, N+1} \) and \( N \in \mathbb{N} \). Any restriction of \( \Delta_L \) is also denoted by the same notation \( \Delta_L \).

**Theorem 3.3.** The Lévy Laplacian \( \Delta_L \) restricted on \( E^\gamma_{-p, N+1} \) is a self-adjoint operator densely defined on \( E^\gamma_{-p, N} \) for each \( 1 \leq \gamma \leq 2, p \geq 1 \) and \( N \in \mathbb{N} \).

**Proof.** We can apply the same proof of Theorem 2 in [27] to this theorem. \( \square \)

Let \( 1 \leq \gamma \leq 2 \). For each \( t \geq 0 \) we consider an operator \( G^\gamma_t \) on \( E^\gamma_{-p, \infty} \) defined by

\[
G^\gamma_t \Phi = \int_{\mathbb{R}} e^{\lambda t} \Phi_{\lambda} \, d\lambda
\]
for $\Phi = \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in E^{-\gamma}_{-p,\infty}$. Then we have the following:

**Theorem 3.4.** For each $\gamma \in [1,2]$ the family $\{G_t^\gamma; t \geq 0\}$ is an equi-continuous semigroup of class $(C_0)$ generated by $|T|^{-\gamma+1} \Delta_L$ as a continuous linear operator defined on $E^{-\gamma}_{-p,\infty}$.

**Proof.** For any $t \geq 0$, $p \geq 1$ and $N \in \mathbb{N}$, the norm $\|G_t^\gamma \Phi\|_{-p,N}$ for $\Phi = \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in E^{-\gamma}_{-p,\infty}$, $\Phi_\lambda \in D(\Lambda)$, $n = 0, 1, 2, \ldots$ can be estimated as follows:

$$
\|G_t^\gamma \Phi\|_{-p,N}^2 = \int_{\mathbb{R}} \left| e^{-i(t(|t|)\gamma)} \Phi_\lambda \right|^2 \alpha_N^\gamma(\lambda) \, d\lambda \\
\leq \int_{\mathbb{R}} \|\Phi_\lambda\|^2 \alpha_N^\gamma(\lambda) \, d\lambda \\
= \|\Phi\|_{-p,N}^2.
$$

Hence the family $\{G_t^\gamma; t \geq 0\}$ is equi-continuous in $t$. It is easily checked that $G_0^\gamma = I, G_t^\gamma G_s^\gamma = G_{t+s}^\gamma$ for each $t, s \geq 0$. We can also estimate that

$$
\|G_t^\gamma \Phi - G_s^\gamma \Phi\|_{-p,N}^2 = \int_{\mathbb{R}} \left| e^{-i(t(|t|)\gamma)} - e^{-i(s(|s|)\gamma)} \right|^2 \|\Phi_\lambda\|^2 \alpha_N^\gamma(\lambda) \, d\lambda \\
\leq 4 \int_{\mathbb{R}} \|\Phi_\lambda\|^2 \alpha_N^\gamma(\lambda) \, d\lambda \\
= 4\|\Phi\|_{-p,N}^2 < \infty
$$

for each $t, s \geq 0, N \in \mathbb{N}$ and $\Phi = \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in E^{-\gamma}_{-p,\infty}$. Therefore, by the Lebesgue dominated convergence theorem, we get

$$
\lim_{t \to t_0} G_t^\gamma \Phi = G_{t_0}^\gamma \Phi \quad \text{in} \quad E^{-\gamma}_{-p,\infty}
$$

for each $t_0 \geq 0$ and $\Phi \in E^{-\gamma}_{-p,\infty}$. Thus the family $\{G_t^\gamma; t \geq 0\}$ is an equi-continuous semigroup of class $(C_0)$. We next prove that the infinitesimal generator of the semigroup is given by $|T|^{-\gamma+1} \Delta_L$. For any $N \in \mathbb{N}$ and $p \geq 1$, we see that

$$
\|G_t^\gamma \Phi - \Phi - |T|^{-\gamma+1} \Delta_L \Phi\|_{-p,N}^2 \\
= \int_{\mathbb{R}} \left| e^{-i(t(|t|)\gamma)} - 1 \Phi_\lambda + \left( \frac{|\Lambda|}{|T|} \right)^\gamma \Phi_\lambda \right|^2 \alpha_N^\gamma(\lambda) \, d\lambda. \quad (3.4)
$$

Since $\Phi = \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in E^{-\gamma}_{-p,\infty}$, we have

$$
\int_{\mathbb{R}} \|\Phi_\lambda\|^2 \alpha_N^\gamma(\lambda) \, d\lambda < \infty. \quad (3.5)
$$

By the mean value theorem, for any $t > 0$ there exists a constant $\theta \in (0,1)$ such that

$$
\left| e^{-i(t(|t|)\gamma)} - 1 \right| = \left( \frac{|\Lambda|}{|T|} \right)^\gamma e^{-i\theta(t(|t|)\gamma)} \leq \left( \frac{|\Lambda|}{|T|} \right)^\gamma.
$$

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Therefore we can estimate each term in (3.4) as follows:

\[
\alpha_N^\gamma(\lambda) \left\| \frac{e^{-t\left(\frac{|\lambda|}{|T|}\right)^\gamma} - 1}{t} \Phi_\lambda + \left( \frac{|\lambda|}{|T|} \right)^\gamma \Phi_\lambda \right\|_{-p}^2
\]

\[
= \frac{\alpha_N^\gamma(\lambda)}{t} \left| \frac{e^{-t\left(\frac{|\lambda|}{|T|}\right)^\gamma} - 1}{t} \right|^2 \left( \frac{|\lambda|}{|T|} \right)^\gamma \left\| \Phi_\lambda \right\|_{-p}^2
\]

\[
\leq 4 \max \left\{ \left| T \right|^{-2(\gamma - 1)} , 1 \right\} \alpha_{N+1}^\gamma(\lambda) \left\| \Phi_\lambda \right\|_{-p}^2.
\]

Note that

\[
\lim_{t \to 0} \left| \frac{e^{-t\left(\frac{|\lambda|}{|T|}\right)^\gamma} - 1}{t} \right|^2 \left( \frac{|\lambda|}{|T|} \right)^\gamma = 0.
\]

Thus by (3.5) we can apply the Lebesgue dominated convergence theorem to obtain

\[
\lim_{t \to 0} \left\| \frac{G_i^\gamma \Phi - \Phi}{t} - \left| T \right|^{-\gamma+1} \Delta_L \Phi \right\|_{-p, N}^2 = 0.
\]

Hence the proof is completed. \(\square\)

**Remark:** For each \(N \in \mathbb{N}\), we can write \(\Delta_L\) and \(G_i^\gamma\) acting on \(E_{-p, \infty}^\gamma\) as

\[
\Delta_L = -\int_{\mathbb{R}} \frac{|\lambda|^\gamma}{|T|} \alpha_N^\gamma(\lambda) \, d\lambda
\]

and

\[
G_i^\gamma = \int_{\mathbb{R}} e^{-t\left(\frac{|\lambda|}{|T|}\right)^\gamma} \alpha_N^\gamma(\lambda) \, d\lambda.
\]

These formulations can be regarded as the diagonalizations of the operators \(\Delta_L\) and \(G_i^\gamma\).

### 4. A stochastic process generated by the Lévy Laplacian

In this section, we will give a stochastic process generated by the extended Lévy Laplacian by considering the stochastic expression of the operator \(G_i^\gamma\).

Let \(\{X_i^\gamma; t \geq 0\}\) be a strictly stable process with the characteristic function of \(X_i^\gamma\) given by

\[
E[e^{i \alpha X_i^\gamma}] = e^{-|\alpha|^\gamma}, \quad 1 \leq \gamma \leq 2.
\]

Take a smooth function \(\eta_T \in E\) with \(\eta_T = \frac{1}{|T|}\) on \(T\). Define an operator \(\widetilde{G}_i^\gamma\) acting on \(S[E_{-p, \infty}^\gamma]\) by

\[
\widetilde{G}_i^\gamma = SG_i^\gamma S^{-1}.
\]
Here the space $S[\mathbb{E}^\gamma_{-p,\infty}]$ is endowed with the topology induced from $\mathbb{E}^\gamma_{-p,\infty}$ by the $S$-transform. Then by Theorem 3.4, $\{\widetilde{G}^\gamma_t; t \geq 0\}$ is an equi-continuous semigroup of class $(C_0)$ generated by the operator $|T|^{-\gamma+1}\widetilde{\Delta}_L$.

Let $\{X^\gamma_t; t \geq 0\}$ be an $E$-valued stochastic process defined by

$$X^\gamma_t = \xi + X^\gamma_T \eta_T, \quad \xi \in E.$$ 

Then we have the following theorem.

**Theorem 4.1.** Let $1 \leq \gamma \leq 2$. Then for all $F \in S[\mathbb{E}^\gamma_{-p,\infty}]$, the following equality holds

$$\widetilde{G}^\gamma_t F(\xi) = \mathbb{E}[F(X^\gamma_t)| X^\gamma_0 = \xi].$$

**Proof.** First consider the case when $F \in S[\mathbb{E}^\gamma_{-p,\infty}]$ is given by

$$F(\xi) = S(J_u[f])(\xi) = \int_{T^n} f(u) \prod_{\nu=1}^n e^{i a_\nu \xi(u_\nu)} du, \quad \sum_{\nu=1}^n a_\nu = \lambda.$$

Then we have

$$\mathbb{E}[F(\xi)| X^\gamma_0 = \xi] = \mathbb{E}[F(\xi + X^\gamma_T \eta_T)]$$

$$= \int_{T^n} f(u) \prod_{\nu=1}^n e^{i a_\nu \xi(u_\nu)} \mathbb{E}[e^{i \sum_{\nu=1}^n a_\nu \eta_\nu}] du$$

$$= e^{-t \mathbb{E}(\sum_{\nu=1}^n a_\nu \eta_\nu)} F(\xi)$$

$$= \widetilde{G}^\gamma_t F(\xi).$$

Next, let $F \in S[\mathbb{E}^\gamma_{-p,\infty}]$ be represented by $F = \int_{\mathbb{R}} F_\lambda d\lambda$ with $F_\lambda$ being expressed in the following form:

$$F_\lambda(\xi) = \lim_{N \to \infty} \sum_{\alpha \in A \lambda} S(J_{\alpha}[f_{\alpha}^{\lambda}](\xi).$$

Hence we have

$$\int_{\mathbb{R}} \mathbb{E} \left[ \left| F_\lambda(\xi + X^\gamma_T \eta_T) \right| \right] \, d\lambda$$

$$= \int_{\mathbb{R}} \mathbb{E} \left[ \lim_{N \to \infty} \left| \sum_{\alpha \in A \lambda} S(J_{\alpha}[f_{\alpha}^{\lambda}](\xi + X^\gamma_T \eta_T) \right| \right] \, d\lambda$$

$$= \int_{\mathbb{R}} \lim_{N \to \infty} \left| \sum_{\alpha \in A \lambda} S(J_{\alpha}[f_{\alpha}^{\lambda}](\xi) \right| \, d\lambda$$

$$= \int_{\mathbb{R}} |F_\lambda(\xi)| \, d\lambda.$$
Since $F_\lambda \in \mathcal{S} [\mathbb{E}^2_{-p,\infty}]$, there exists some $\Phi_\lambda \in \mathbb{E}^2_{-p,\infty}$ such that $F_\lambda (\xi) = \mathcal{S}[\Phi_\lambda] (\xi) = \langle \Phi_\lambda, \varphi_\xi \rangle$ for any $\xi \in E$ and $\lambda \in \mathbb{R}$. By the Schwarz inequality, we see that

$$
\int_{\mathbb{R}} |F_\lambda (\xi)| \, d\lambda \leq \int_{\mathbb{R}} \|\Phi_\lambda\|_p \|\varphi_\xi\|_p \, d\lambda \\
\leq \left\{ \int_{\mathbb{R}} \alpha_M^\gamma (\lambda)^{-1} \, d\lambda \right\}^{1/2} \left\{ \int_{\mathbb{R}} \|\Phi_\lambda\|_p^2 \alpha_M^\gamma (\lambda) \, d\lambda \right\}^{1/2} \|\varphi_\xi\|_p
$$

< \infty,

for all $\xi \in E$ and some $M \geq 1$, where $\varphi_\xi (x) = : e^{(\alpha, \xi)} :$. Therefore by the continuity of $\widetilde{G}_t^\gamma$ we get

$$
E[F(\xi + X_t^\gamma \eta)] = \int_{\mathbb{R}} E[F_\lambda (\xi + X_t^\gamma \eta)] \, d\lambda \\
= \int_{\mathbb{R}} \widetilde{G}_t^\gamma F_\lambda (\xi) \, d\lambda \\
= \widetilde{G}_t^\gamma F(\xi).
$$

Thus we obtain the assertion. \qed

Theorem 4.1 says that the infinite dimensional stochastic process \{\!\!X_t^\gamma; t \geq 0\!\!\} is generated by $|T|^{-\gamma+1} \Delta_L$ defined on $\mathbb{E}^2_{-p,\infty}$ for each $1 \leq \gamma \leq 2$.

For any $\Phi \in (E)^*$ and $\eta \in E$, we define the translation operator $\tau_\eta$ on $(E)^*$ by $S[\tau_\eta \Phi] (\xi) = S[\Phi] (\xi + \eta)$, $\xi \in E^c$ (See [14].) Then we can restate Theorem 4.1 in words of generalized white noise functionals.

**Corollary 4.2.** Let $1 \leq \gamma \leq 2$. Then for all $\Phi \in \mathbb{E}^2_{-p,\infty}$, the following equality holds

$$
\widetilde{G}_t^\gamma \Phi = E \left[ \tau_{X_t^\gamma \eta} \Phi \right].
$$

By Corollary 4.2 we can see that \{\!\!\tau_{X_t^\gamma \eta}; t \geq 0\!\!\} is an operator-valued stochastic process and \{\!\!E[\tau_{X_t^\gamma \eta}]; t \geq 0\!\!\} is an equi-continuous semigroup of class $(C_0)$ generated by $|T|^{-\gamma+1} \Delta_L$ defined on $\mathbb{E}^2_{-p,\infty}$ for each $1 \leq \gamma \leq 2$.

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