Connections over Two-Dimensional Cell Complexes
A preliminary account

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1 The cell complex

We shall work with a cell complex. Here we recall the definitions.

Let $X$ be a topological space. Let $B_k$ be the unit–ball in $\mathbb{R}^k$, with $B_0$ being just a one–point set. By a $k$–cell we shall mean a continuous map $b : B_k \to X$ which is a homeomorphism in the interior of $B_k$. Two $k$–cells will be considered equivalent, forming an unoriented $k$-cell, if they differ by an orientation–preserving homeomorphism in the interior of $B_k$. We shall also use oriented $k$–cells, with the obvious meaning. A based $k$–cell is a $k$–cell with a chosen basepoint on the boundary of $B_k$. If $b$ is a $k$–cell we shall call $b(B_k^0)$ the interior of $b$.

By a two–dimensional cell complex we mean a set $X$ along with a family of “attachment maps” from a finite non–empty set of based $k$–cells to $X$, for $k = 0,1,2$. The attachment maps will often also be called cells. Zero–cells shall typically be denoted $v_i$, 1–cells as $e_\alpha$, and 2–cells by $\sigma$. The defining conditions for a cell complex will be:

- distinct cells have disjoint interiors
• if $\sigma$ is an oriented two-cell, with base point $v_0 \in \partial B_2$ then there is a
sequence of points $v_0, v_1, ..., v_{N-1}, v_N = v_0$, going around $\partial B_2$ in the
positive sense, such that each $\sigma|[v_{i-1}, v_i]$ is a 1-cell

Furthermore, we shall assume that there is no boundary for the complex, i.e.

• each 0–cell is on the boundary of a 1–cell, and each 1–cell is on the
  boundary of a 2–cell.

We shall say that the complex is oriented if it is possible to choose an
orientation on each 2–cell in such a way that

• if a 1–cell lies on the boundary of two 2–cells then these cells induce
  opposite orientations on the 1–cell (in particular, each 1–cell lies on the
  boundary of at most two 2–cells)

• a 1–cell can appear at most twice on the boundary of a 2–cell, and if
  it does appear twice on such a boundary then the induced orientations
  are opposite to each other.

2 Connections

We shall define the notions of ‘connection’ and ‘gauge transformations’ over
a cell-complex In this section $C$ is a cell complex and $G$ a Lie group, with Lie
algebra denoted $LG$.

A connection, with values in the group $G$, over the complex $C$ is a map $x$
from the set of oriented 1–cells to $G$, such that for any 1–cell $e$

$$x(\overline{e}) = x(e)^{-1}$$

where $\overline{e}$ is the orientation reverse of $e$.

If $\kappa$ is a “curve” in the complex, i.e. a sequence of oriented 1–cells
$e_1, ..., e_N$, each ending where the next begins, then we write

$$x(\kappa) = x(e_N)...x(e_1)$$

Now let $x$ be a connection and $\sigma$ a based, oriented 2–cell. We will call
$x(\partial \sigma)$ the curvature of the connection $x$ over $\sigma$, and denote it by $K_x(\sigma)$

$$K_x(\sigma) = x(\partial \sigma) \quad (1)$$

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when the basepoint of \( \sigma \) is changed, \( K_x(\sigma) \) is conjugated.

The set of all connections, with values in a fixed group \( G \), over \( C \) will be denoted \( A_C \).

A connection \( x \) is flat if \( x(\partial \sigma) = e \), the identity in \( G \), for every two-cell \( \sigma \).

A gauge transformation is a map \( \theta : \{ \text{all 0–cells} \} \to G \). This is a group, to be denoted \( G_C \), under pointwise multiplication.

The group \( G_C \) acts on \( A_C \) as follows:

\[
x^\theta(ab) = \theta(b)^{-1}x(ab)\theta(a)
\]

for any oriented 1–cell \( ab \), running from \( a \) to \( b \).

3 The chain complex associated to a flat connection

In this section we shall describe a chain complex associated with a flat connection over a cell-complex.

If \( x \) is a connection then we have the tangent space \( T_x A_C \), consisting of all maps

\[
A : \{1–cells\} \to LG
\]

such that

\[
A(\overline{e}) = -(\text{Ad } x(e))A(e)
\]

for every oriented one-cell \( e \). We shall also denote \( T_x A_C \) by

\[
C^1_x = T_x A_C
\]

Assume now that \( x \) is a flat connection. Let \( C^2_x \) be the set of all maps

\[
f : \{2–cells\} \to LG
\]

such that

\[
f(\overline{e}) = -f(e)
\]

and if \( \sigma_a \) is an oriented cell based at \( a \) and \( \sigma_b \) the same cell, based now at \( b \), then

\[
f(\sigma_b) = \text{Ad } x(ab) f(\sigma_a)
\]
where $ab$ is the curve formed by the sequence of 1-cells on $\partial\sigma_a$ running positively from $a$ to $b$. Because $x(\partial\sigma) = e$, the identity, the condition defining $f$ is self-consistent.

Finally, let

$$C^0_x = \{\text{all maps } H: \{\text{all 0-cells}\} \to LG\}$$

(5)

It is useful to note that this may be viewed as the Lie algebra of the group of gauge transformations:

$$LG_{C} = C^0_x$$

(6)

The orbit map through $x \in A$,

$$\gamma_x: G_{C} \to A_{C}: \theta \mapsto x^\theta$$

has the derivative

$$\gamma'_x: LG_{C} \to T_xA_{C}$$

given by

$$\gamma'_x(Y)(ab) = Y(a) - (\text{Ad}x(ab)^{-1})Y(b)$$

(7)

for every 1-cell $ab$ running from $a$ to $b$.

The map $\gamma'_x$ will also be denoted $D^0_\xi$:

$$D^0_\xi = \gamma'_x : C^0_x \to C^1_x$$

(8)

If $\sigma$ is a based oriented 2-cell then we have the derivative of the map $x \mapsto K_x(\sigma)$, left translated back to the identity in $G$:

$$dK_x(\sigma): T_xA \to LG$$

$$A \mapsto A(e_1) + \text{Ad}x(e_1)^{-1}A(e_2) + \cdots + \text{Ad}(x(e_{N-1})\cdots x(e_1))^{-1}A(e_N)$$

(9)

where $\partial\sigma = e_N\cdots e_1$. Sometimes, however, it will be convenient to take the derivative as a map $dK_x(\sigma): T_xA \to T_{K_x(\sigma)}G$, and write the map in (9) as $K_x(\sigma)^{-1}dK_x(\sigma)$.

If $x$ is flat then, as is readily verified, $dK_x(\sigma)$ is independent of the choice of basepoint.

For flat $x$ we define

$$D^1_\xi : C^1_x \to C^2_x : A \mapsto D^1_\xi A$$

(10)

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where $D^x_i A$ is the element of $C^2_x$ specified by
\[ D^x_i A(\sigma) = dK_x(\sigma)A \] (11)
for each based oriented 2–cell $\sigma$.
Our first observation is

**Lemma 1** For any flat connection $x$,
\[ D^x_i D^0_0 = 0 \]

**Proof** Let $H \in C^0_x$ and $\sigma$ a based oriented 2–cell. Write $\partial \sigma = e_N \ldots e_1$, where the $e_i$ are oriented one–cells, and $e_i$ runs from $v_{i-1} v_i$. Then
\begin{align*}
D^x_i D^0_0 H(\sigma) &= \sum_{i=1}^N \text{Ad}(x(e_{i-1}) \ldots x(e_1))^{-1}[H(v_{i-1}) - \text{Ad}(x(e_i)^{-1})H(v_i)] \\
&= H(v_0) - \text{Ad}(K_x(\sigma)^{-1})H(v_0) \\
&= 0
\end{align*}
the last equality following from $K_x(\sigma) = x(\partial \sigma) = e$, the identity in $G$, because $x$ is flat.

Thus we obtain a chain complex:
\[ 0 \to C^0_x \to C^1_x \to C^2_x \to 0 \] (12)
where the first differential is $D^0_0$ and the second $D^x_i$.

The zero–th cohomology is
\[ H^0(C_x) = \ker D^x_0 \]

**Proposition 1** Suppose that, for each 0–cell $a$, $Z = 0$ is the only vector in $LG$ which satisfies $\text{Ad}(x(L))Z = Z$ for every loop $L$ in $C$ based at $a$. Then
\[ H^0(C_x) = \{0\} \]

**Proof** Suppose $Y \in \ker D^x_0$ and $a$ is any 0–cell. By our hypotheses on the cell complex $C$, $a$ is the base point of some 1–cell $e_1$, and $e_1$ is part of the boundary of some 2–cell. So, in particular, there exist loops with base point $a$. 

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If \( L = e_N \ldots e_1 \) is any loop based at \( a \), with the one-cell \( e_i \) running from vertex \( a_{i-1} \) to vertex \( a_i \), then the definition of \( D_0^e \) shows that

\[
Y(a_{i-1}) = \text{Ad}(x(e_i))^{-1}Y(a_i)
\]

holding for \( i = 1, \ldots, N \). Combining these we have, using \( a = a_0 \),

\[
Y(a) = \text{Ad}(x(L))Y(a)
\]

Since this holds for every loop \( L \) based at \( a \), it follows by the hypothesis that \( Y(a) = 0 \). Since \( a \) is any arbitrary 0-cell of the complex, we have \( Y = 0 \). \[QED\]

The first cohomology group \( H^1(C_x) \) is:

\[
H^1(C_x) = \ker D_1^x / \text{Im} D_0^x
\] (13)

Define the tangent space

\[
T_xA^0_C
\] (14)

to be the set of all vectors in \( T_xA_C \) which are initial tangents to \( C^\infty \) paths, lying entirely on \( A^0_C \subset A_C \).

If \( x \) is a flat connection then \( K_x(\sigma) = e \) for every 2-cell \( \sigma \) and so

\[
T_xA^0_C \subset \ker D_1^x
\] (15)

We shall say that

a flat connection \( x \) is regular if \( T_xA^0_C = \ker D_1^x \) (16)

The following result is now clear:

**Proposition 2** For any regular flat connection \( x \),

\[
T_xA^0_C / \gamma_x^1 (L\mathcal{G}_C) = H^1(C_x)
\]

4 The 2–form \( \Omega \)

In this section, \( \langle \cdot, \cdot \rangle \) is a non–degenerate bilinear form on the Lie algebra \( L\mathcal{G} \) of \( G \). Using this we shall construct and study a special 2–form \( \Omega \) on the space of flat connections.
Consider $A, B \in T_x \mathcal{C}$. Let $\sigma$ be a based, oriented, two-cell whose boundary $\partial \sigma$ is $e_N \ldots e_1$, where $e_i$ is the one-cell from $a_{i-1}$ to $a_i$. Define

$$
\Omega_\sigma(A, B)_x = \Omega_x(A, B)_\sigma
$$

(17)

$$
= \frac{1}{2} \sum_{1 \leq i, j \leq N} \epsilon_{ij} \langle f_{i-1}^{-1} A(e_i), f_{j-1}^{-1} B(e_j) \rangle
$$

where $\epsilon_{ij}$ is 1 if $i < j$, is $-1$ if $i > j$, and is 0 if $i = j$, and

$$
f_i = \text{Ad}(x(e_{i-1})\ldots x(e_1))
$$

(18)

with $f_0$ being the identity map on $LG$.

In general, the value $\Omega_x(A, B)_\sigma$ will depend on the choice of basepoint on $\sigma$. However, for flat connections we have:

**Lemma 2** Let $x$ be a flat connection. If $A, B \in \ker D^*_1$ then $\Omega_x(A, B)_{\sigma}$ is independent of the choice of basepoint of the oriented 2-cell $\sigma$.

**Proof** Let $\partial \sigma = e_N \ldots e_1$, with $e_i$ running from $v_{i-1}$ to $v_i$, and $v_N = v_0$ is the basepoint. Moving the basepoint over to $v_1$, we call the 2-cell $\sigma'$. So $\partial \sigma' = e_1e_N \ldots e_2$.

The terms in $\Omega_x(A, B)_\sigma$ and $\Omega_x(A, B)_{\sigma'}$ are identical except for those which involve $A(e_1)$ and those which involve $B(e_1)$. With this in mind we write,

$$
\Omega_x(A, B)_{\sigma} = \frac{1}{2} \sum_{i,j} \epsilon_{ij} \langle f_{i-1}^{-1} A(e_i), f_{j-1}^{-1} B(e_j) \rangle
$$

$$
= \frac{1}{2} \langle A(e_1), L \rangle
$$

where

$$
L = \sum_{j=2}^{N} f_{j-1}^{-1} B(e_j) + \frac{1}{2} \sum_{i=2}^{N} \langle f_{i-1}^{-1} A(e_i), \sum_{j=i}^{N} f_{j-1}^{-1} B(e_j) \rangle - \sum_{j=1}^{i-1} f_{j-1}^{-1} B(e_j)
$$

(19)

Using $\text{Ad}(x(e_i)\ldots x(e_2))^{-1} = \text{Ad}(x(e_1))f_i^{-1}$, we also have

$$
\Omega_x(A, B)_{\sigma'} = -\frac{1}{2} \langle \text{Ad} x(e_1) f_N^{-1} A(e_1), \text{Ad} x(e_1) \sum_{j=2}^{N} f_{j-1}^{-1} B(e_j) \rangle
$$

$$
+ \frac{1}{2} \sum_{i=2}^{N} \langle \text{Ad} x(e_1) f_{i-1}^{-1} A(e_i), \text{Ad} x(e_1) M \rangle
$$

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where

$$M = \sum_{j=i+1}^{N} f_{j-1}^{-1} B(e_j) + f_{N-1}^{-1} B(e_1) - \sum_{j=2}^{i-1} f_{j-1}^{-1} B(e_j) \quad (20)$$

Then

$$2\Omega_x(A, B)_\sigma - 2\Omega_x(A, B)_{\sigma'} = \langle (1 + f_N^{-1}) A(e_1), \sum_{j=2}^{N} f_{j-1}^{-1} B(e_j) \rangle$$

$$+ \sum_{i=2}^{N} \langle f_{i-1}^{-1} A(e_i), -B(e_1) - f_{N}^{-1} B(e_1) \rangle$$

$$= \langle (1 + f_N^{-1}) A(e_1), \sum_{j=1}^{N} f_{j-1}^{-1} B(e_j) - B(e_1) \rangle$$

$$- \langle \sum_{i=1}^{N} f_{i-1}^{-1} A(e_i) - A(e_1), (1 + f_N^{-1}) B(e_1) \rangle$$

$$= \langle (f_N - f_N^{-1}) A(e_1), B(e_1) \rangle + \langle (1 + f_N^{-1}) A(e_1), \sum_{j=1}^{N} f_{j-1}^{-1} B(e_j) \rangle$$

$$- \langle \sum_{i=1}^{N} f_{i-1}^{-1} A(e_i), (1 + f_N^{-1}) B(e_1) \rangle$$

$$= \langle (f_N - f_N^{-1}) A(e_1), B(e_1) \rangle$$

$$+ \langle (1 + f_N^{-1}) A(e_1), dK_{\sigma}(x) B \rangle$$

$$- \langle (1 + f_N^{-1}) B(e_1), dK_{\sigma}(x) A \rangle$$

In this, using $f_N = \text{Identity}$ (because $x$ is flat), the first term is 0, and the hypothesis that $A, B \in \ker D^2_1$ shows that the remaining terms also equal zero. [QED]

Suppose $C$ is oriented. Then each 2-cell comes with a favored orientation. Now choose, arbitrarily, a basepoint for each 2-cell. Define

$$\Omega_x(A, B) = \sum_{\text{all 2-cells } \sigma} \Omega_x(A, B)_\sigma \quad (21)$$

where the sum is over all positively oriented 2-cells, with basepoints as chosen.

Clearly $\Omega_x$ is skew-symmetric, and so $\Omega$ is a 2-form on $A_C$.  

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5 \(\Omega\) is closed

Let \(\sigma\) be a based oriented 2-cell with boundary \(\partial\sigma = e_N...e_1\). Consider the map

\[
\Phi_\sigma : \mathcal{C} \rightarrow G^N
\]

given by

\[
\Phi_\sigma(x) = (x(e_1), x(e_2)x(e_1), ..., x(e_N)...x(e_1))
\]

Now let \(\omega\) be the 2–form on \(G^N\) given by

\[
\omega_x(xA, xB) = \sum_{i,j} \epsilon_{ij} (A_i - A_{i-1}, B_i - B_{i-1})
\]

where \(A, B \in (LG)^N\). Then

\[
\frac{1}{2} \Phi_\sigma^* \omega = \Omega_\sigma
\]

and so

\[
\Omega = \sum_\sigma \frac{1}{2} \Phi_\sigma^* \omega
\]

where the sum is over all 2–cells, each taken once only, with specified basepoint and orientation as specified by the orientation of the cell complex \(\mathcal{C}\).

Using this we prove:

**Theorem 1** The 2–form \(\Omega\) is closed:

\[
d\Omega = 0
\]

**Proof.** From (23) and using the expression for \(d\omega\) given in Lemma 3 below, we have

\[
d\Omega_x(A, B, C) = -\frac{1}{2} \sum_\sigma \sum_i \langle A(e_i), [B(e_i), C(e_i)] \rangle
\]

where the sum \(\sum_i\) is over the oriented 1–cells on the boundary of \(\sigma\).

Now each oriented 1–cell \(e_i\) appears in the sum on the right in (25) exactly once. If \(e\) appears on the boundary of some \(\sigma\) then there is a \(\sigma'\) (possibly the same as \(\sigma\)) such that the orientation reverse \(\overline{\sigma}\) appears on the boundary of \(\sigma'\). Moreover,

\[
A(\overline{\sigma}) = -\text{Ad}(x(e))A(e)
\]

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and similarly for \(B(\bar{e})\) and \(C(\bar{e})\). Since
\[
\langle -\text{Ad} \ (x(e))A(e), \left[-\text{Ad} \ (x(e))B(e), -\text{Ad} \ (x(e))C(e)\right]\rangle = -\langle A(e), [B(e), C(e)] \rangle
\]
it follows that the terms in the sum on the right in (25) cancel in pairs, leaving a sum of 0.

We have used the following result (Proposition E of [8]) for which we include a proof for the reader’s convenience.

**Lemma 3** Let \(\omega\) be the 2–form on \(G^N\) given by
\[
\omega_x(xA, xB) = \sum_{i,j} \epsilon_{ij} \langle A_i - A_{i-1}, B_i - B_{i-1} \rangle
\]
where \(A, B \in (LG)^N\). Then
\[
\omega_x(xA, xB) = \sum_{i=1}^{N} \left( \langle A_{i-1}, B_i \rangle - \langle A_i, B_{i-1} \rangle \right)
\]
and
\[
d\omega = -\sum_{i=1}^{N+1} \langle A_i - A_{i-1}, [B_i - B_{i-1}, C_i - C_{i-1}] \rangle
\]
where, for any vector \(X = (X_1, ..., X_N) \in (LG)^N\) we take \(X_i = 0\) for \(i \leq 0\) and \(i > N\).

**Proof.** We have
\[
\omega_x(xA, xB) = \sum_{i=1}^{N} \langle A_i - A_{i-1}, B_N - B_i - (B_{i-1} - B_0) \rangle
\]
\[
= \sum_{i=1}^{N} \left( \langle A_{i-1}, B_i \rangle - \langle A_i, B_{i-1} \rangle \right) + \sum_{i=1}^{N} \left( \langle A_{i-1}, B_{i-1} \rangle - \langle A_i, B_i \rangle \right)
\]
\[
+ \langle A_N - A_0, B_N \rangle + \langle A_N - A_0, B_0 \rangle
\]
\[
= \sum_{i=1}^{N} \left( \langle A_{i-1}, B_i \rangle - \langle A_i, B_{i-1} \rangle \right)
\]

The 2–form \(\omega\) is manifestly left–invariant. So, instead of writing \(\omega_x(xA, xB)\) we shall simply write \(\omega(A, B)\). Next we compute \(d\omega\). By left invariance it suffices to compute it at the identity in \(G\):
\[
d\omega(A, B, C) = A \left( \omega(B, C) \right) + C \left( \omega(A, B) \right) + B \left( \omega(C, A) \right)
\]
\[
+ \omega(A, [B, C]) + \omega(C, [A, B]) + \omega(B, [C, A])
\]
The first three terms, being derivatives of constant functions, are each equal to zero. Proceeding with the remaining terms we have:

\[ d\omega(A, B, C) = \sum_{i=1}^{N} \left( \langle A_{i-1}, [B_i, C_i] - \langle A_i, [B_{i-1}, C_{i-1}] \rangle \right) \]

\[ + \sum_{i=1}^{N} \left( \langle C_{i-1}, [A_i, B_i] - \langle C_i, [A_{i-1}, B_{i-1}] \rangle \right) \]

\[ + \sum_{i=1}^{N} \left( \langle B_{i-1}, [C_i, A_i] - \langle B_i, [C_{i-1}, A_{i-1}] \rangle \right) \]

\[ = \sum_{i=1}^{N} \langle A_i, [B_{i+1}, C_{i+1}] - [B_{i-1}, C_{i-1}] \rangle \]

\[ + \sum_{i=1}^{N} \langle A_i, [B_i, C_i] - [B_{i-1}, C_{i-1}] \rangle \]

\[ + \sum_{i=1}^{N} \langle A_i, [B_{i-1}, C_i] - [B_{i+1}, C_i] \rangle \]

\[ = \sum_{i=1}^{N} \langle A_i, S_i \rangle \]

where

\[ S_i = [B_{i+1}, C_{i+1}] - [B_{i-1}, C_{i-1}] + [B_i, C_i] - [B_{i-1}, C_{i-1}] + [B_i, C_{i+1}] - [B_{i+1}, C_i] \]

\[ = [B_{i+1} - B_i, C_{i+1} - C_i] - [B_i - B_{i-1}, C_i - C_{i-1}] \]

Thus

\[ d\omega(A, B, C) = \sum_{i=1}^{N+1} \langle A_{i-1}, [B_i - B_{i-1}, C_i - C_{i-1}] \rangle \]

\[ = \sum_{i=1}^{N+1} \langle A_{i-1}, [B_i - B_{i-1}, C_i - C_{i-1}] \rangle \]

\[ + \sum_{i=1}^{N+1} \langle A_i, [B_i - B_{i-1}, C_i - C_{i-1}] \rangle \]

\[ = - \sum_{i=1}^{N+1} \langle A_i - A_{i-1}, [B_i - B_{i-1}, C_i - C_{i-1}] \rangle \]

as claimed. QED
6 A moment–map type property

Let us briefly review the situation. Associated to our cell complex \( \mathcal{C} \) we have the space \( \mathcal{A}_c \) of connections, on which there is a 2–form \( \Omega \), and the group \( \mathcal{G}_c \) of gauge transformations acts on \( \mathcal{A}_c \). In this section we shall derive a formula which is strongly reminiscent of the definition of a moment map in symplectic mechanics. The formula of this section originated in a result proven in [8].

Let \( x \) be a connection. Then we have the orbit map

\[
D^x_0 = \gamma'_x : LG_c \to T_x \mathcal{A}_c
\]

We prove

**Theorem 2** Assume that \( \mathcal{C} \) is oriented. Let \( x \in \mathcal{A}_c \) be a connection over \( \mathcal{C} \), \( A \) any element of \( \mathcal{C}_x^1 \), and \( H \) any element of \( \mathcal{C}_x^0 \). Then:

(a) the following holds:

\[
\Omega_x(D^0_xH, A) = \sum_{\sigma} \left\langle \frac{1}{2} (1 + \text{Ad}(K_\sigma(x))^{-1}) H(v_\sigma), dK_\sigma(x)A \right\rangle
\]

(26)

where \( v_\sigma \) is the chosen baspoint on the boundary of the 2–cell \( \sigma \), and the sum is over all oriented 2–cells

(b) if \( x \) is a flat connection then

\[
\Omega_x(D^x_1H, A) = \langle \overline{H}, D^x_2A \rangle
\]

(27)

where \( \overline{H} \) is the element of \( \mathcal{C}_x^2 \) which associates to each based oriented 2–cell \( \sigma \) the element \( H(v_\sigma) \), with \( v_\sigma \) being the baspoint on \( \sigma \).

(c) if \( x \) is a flat connection and \( A \in \ker D^x_2 \) then

\[
\Omega_x(D^x_1H, A) = 0
\]

(28)

**Proof.** Let \( \sigma \) be a based oriented 2-cell, with \( \partial \sigma = e_N \ldots e_1 \), where \( e_i \) runs from a vertex \( v_{i-1} \) to a vertex \( v_i \). Thus \( v_N = v_0 \) is the base point of \( \sigma \).
We have then
\[
\Omega_x(D^0_x H, A)_{\sigma} = \frac{1}{2} \sum_{i,j} e_{ij} \langle f^{-1}_{i-1} (H(v_{i-1}) - \text{Ad } x(e_i)^{-1} H(v_i)), f^{-1}_{j-1} A(e_j) \rangle
\]
\[
= \frac{1}{2} \sum_{j=1}^N \langle H(v_0) - f^{-1}_{j-1} H(v_{j-1}) - f^{-1}_{j} H(v_j) + f^{-1}_{N} H(v_N), f^{-1}_{j-1} A(e_j) \rangle
\]
\[
= \frac{1}{2} \sum_{j=1}^N \langle (1 + f^{-1}_N) H(v_0) - f^{-1}_{j} H(v_j) - f^{-1}_{j-1} H(v_{j-1}), f^{-1}_{j-1} A(e_j) \rangle
\]
\[
= \langle \frac{1}{2} (1 + K_\sigma(x)^{-1}) H(v_\sigma), dK_\sigma(x) A \rangle - M_\sigma
\]

where $v_\sigma$ is the chosen basepoint $v_0$ on the boundary of the 2-cell $\sigma$, and

\[
M_\sigma = \frac{1}{2} \sum_{j=1}^N \langle \text{Ad } x(e_j)^{-1} H(v_j) + H(v_{j-1}), A(e_j) \rangle
\]

By our assumptions concerning the oriented cell-complex $C$, each 1-cell appears exactly twice in the sum

\[
\sum_\sigma M_\sigma,
\]

once in the form $e$ as a boundary of an oriented 2-cell and once as $\overline{e}$ of either the same 2-cell or a neighboring 2-cell. Suppose $e$ runs from $a$ to $b$; then the terms involving $e$ are:

\[
\langle \text{Ad } x(e)^{-1} H(b) + H(a), A(e) \rangle + \langle \text{Ad } x(\overline{e})^{-1} H(a) + H(b), A(\overline{e}) \rangle
\]
\[
= \langle \text{Ad } x(e)^{-1} H(b) + H(a), A(e) \rangle + \langle \text{Ad } x(e) H(a) + H(b), -\text{Ad } x(e) A(e) \rangle
\]
\[
= 0
\]

Using this we conclude that

\[
\Omega_x(D^0_x H, A) = \sum_\sigma \langle \frac{1}{2} (1 + K_\sigma(x)^{-1}) H(v_\sigma), dK_\sigma(x) A \rangle
\]

where $v_\sigma$ is the chosen basepoint on the boundary of the 2-cell $\sigma$, and the sum is over all oriented 2-cells. This proves part (a). Parts (b) and (c) are immediate consequences. [QED]
7 The continuum version

We now turn to the theory of connections in the continuum.

7.1 The space \( \mathcal{A} \) and the group \( \mathcal{G} \)

We work with a compact oriented two-dimensional manifold \( \Sigma \), a compact, connected Lie group \( G \) whose Lie algebra is equipped with an Ad-invariant metric, and a principal \( G \)-bundle \( \pi : P \to \Sigma \). In particular, the group \( G \) acts freely on \( P \) by a smooth map

\[
P \times G \to P : (p,g) \mapsto pg = R_gp
\]

If \( v \in T_pP \) and \( g \in G \) then we write \( vg \) to mean the vector \( (dR_g)_p v \in T_{R_gp} P \).

If \( X \in LG \) and \( p \in P \) then we define

\[
X^*(p) = \frac{d}{dt} \bigg|_{t=0} p \exp(tX)
\]

Then \( X^* \) is a smooth vector field on \( P \).

A vector \( v \in T_pP \) is called vertical if \( d\pi_p v = 0 \).

A connection \( \omega \) on \( P \) is an \( LG \)-valued 1-form \( \omega \) on \( P \) satisfying two conditions: (i) \( \omega(X^*(p)) = X \) for every \( X \in LG \), (ii) \( R_g^* \omega = \text{Ad}(g^{-1})\omega \) for every \( g \in G \).

Let \( \mathcal{A} \) be the set of all connections on \( P \). This forms an infinite-dimensional affine space under pointwise operations. The tangent space at any \( \omega \in \mathcal{A} \) is

\[
T_\omega \mathcal{A} = \{ \omega' - \omega : \omega' \in \mathcal{A} \}
\]

Denote by \( \mathcal{G} \) the set of all bundle equivalences of \( P \) (gauge transformations), i.e. all \( G \)-equivariant diffeomorphisms \( \phi : P \to P \) for which \( \pi \circ \phi = \pi \). Then \( \mathcal{G} \) is a group under composition.

There is another useful incarnation of \( \mathcal{G} \): it is \( C_G(P,G) \) the set of all smooth maps \( \hat{\phi} : P \to G \) satisfying \( \hat{\phi}(pg) = g^{-1}\hat{\phi}(p)g \) for all \( p \in P \) and \( g \in G \). This forms a group under pointwise multiplication, and there is an isomorphism \( C_G(P,G) \to \mathcal{G} : \hat{\phi} \mapsto \phi \) specified by requiring that for every \( p \in P \), \( \phi(p) = p\hat{\phi}(p) \).

The group \( \mathcal{G} \) acts on \( \mathcal{A} \) by pullbacks:

\[
\mathcal{A} \times \mathcal{G} \to \mathcal{A} : (\omega, \phi) \mapsto \gamma_\omega(\phi) = \phi^* \omega = \text{Ad}(\hat{\phi}^{-1})\omega + \hat{\phi}^{-1} d\hat{\phi}
\]
7.2 The tangent space $T^\omega A$ and the Lie algebra $LG$

We denote by $\mathcal{X}^k(P; LG)$ the linear space of all smooth $LG$-valued $k$-forms $A$ on $P$ satisfying two conditions: (i) $A(v_1, ..., v_k) = 0$ if at least one of the $v_i$ is vertical; (ii) $R^*_g A = \text{Ad}(g^{-1})A$ for every $g \in G$, i.e. $A(v_1g, ..., v_kg) = \text{Ad}(g^{-1})A(v_1, ..., v_k)$ for every $g \in G$, every $p \in P$ and every $v_1, ..., v_k \in T_p P$.

The linear space $\mathcal{X}^0(P; g)$ is, pointwise, a Lie algebra. Considering the incarnation $C_G(P, G)$ of $G$, it is seen that it is reasonable to think of $\mathcal{X}^0(P; g)$ as the Lie algebra of $G$, and so we set

$$LG = \mathcal{X}^0(P; g)$$

There is an exponential map. If $H \in LG$ then we define $e^H \in \hat{G}$ by

$$e^H(p) = \exp(H(p)) \in G$$

(29)

Thus,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} e^{tH}(p) = H(p)$$

It is also readily verified that the tangent space $T^\omega A$ is $\mathcal{X}^1(P; g)$, for any $\omega \in A$:

$$T^\omega A = \mathcal{X}^1(P; g)$$

We will say that a map

$$[0, 1] \to A : t \mapsto \omega_t$$

is a pointwise smooth path in $A$ if $[0, 1] \times P \to T^*P \otimes LG : (t, p) \mapsto \omega_t(p)$ is smooth. For such a path we have the initial tangent vector

$$\left. \frac{d}{dt} \right|_{t=0} \omega_t \in T_{\omega_0} A$$

which is the $LG$-valued 1–form whose value at any $p \in P$ is $\left. \frac{d}{dt} \right|_{t=0} \omega_t(p)$.

Similarly, we have the derivative of the orbit map $\gamma_\omega$;

$$\gamma_\omega' H = \left. \frac{d}{dt} \right|_{t=0} \gamma_\omega(e^{tH}) = dH + [\omega, H] = D^\omega_0 H$$

(30)
7.3 The covariant derivative $D^\omega$ and the curvature $\Omega^\omega$

For a connection $\omega$, and any $LG$-valued $k$–form $\eta$ on $P$, is the covariant derivative of $\eta$ is the $(k + 1)$–form on $P$ given by

$$D^\omega \eta = d\eta + [\omega \wedge \eta]$$

The curvature of $\omega$ is the $LG$–valued 2–form $J(\omega) = \Omega^\omega$ given by

$$J(\omega) = \Omega^\omega = D^\omega \omega$$

and this is in $\overline{\mathcal{X}}^2(P; g)$.

A connection is flat if its curvature is zero.

A simple calculation produces the derivative

$$J'(\omega)A = \frac{d}{dt} \bigg|_{t=0} J(\omega + tA) = dA + [\omega \wedge A] = D^\omega A$$

(31)

which shows that $J'(\omega) : T_\omega A \rightarrow LG$ is in fact the linear map $D^\omega_1$.

Restricting the covariant derivative to $\overline{\mathcal{X}}^k(P; g)$ we have

$$D^\omega = D^\omega_k : \overline{\mathcal{X}}^k(P; g) \rightarrow \overline{\mathcal{X}}^{k+1}(P; g) : \eta \mapsto d\eta + \omega \wedge \eta$$

A calculation shows that

$$(D^\omega)^2 \eta = \Omega^\omega \wedge \eta$$

for every $\eta \in \overline{\mathcal{X}}^k(P; g)$.

7.4 The chain complex $C_\omega$

If $\omega$ is flat then $(D^\omega)^2 = 0$, and so there is a chain complex $C_\omega$:

$$0 \longrightarrow C^0_\omega \xrightarrow{D^\omega_0} C^1_\omega \xrightarrow{D^\omega_1} C^2_\omega \xrightarrow{D^\omega_2} 0$$

(32)

where

$$C^0_\omega = LG, \quad C^1_\omega = T_\omega A, \quad C^2_\omega = \overline{\mathcal{X}}^2(P; g)$$

(33)

If $\omega$ is a flat connection and $A \in T_\omega A$ is the initial tangent to a pointwise smooth path lying entirely on $A^0$, then we say that $A$ is tangent to $A^0$ at $\omega$. The set of all such vectors will be denoted

$$T_\omega \cdot A^0$$

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It is not being claimed that this is a linear space.

Since $J'(\omega) = D^\omega_1$, we have

$$ T_\omega A^0 \subset \ker D^\omega_1 $$

Assume that $\omega$ is regular/irreducible enough that $D^\omega_1$ is surjective/ ker $D^\omega_0 = 0$, and $T_\omega A^0 = \ker D^\omega_1$. Then the cohomology of $C_\omega$ is contained in the first cohomology group

$$ H^1(C_\omega) = \ker D^\omega_1 / \text{Im} D^\omega_0 = T_\omega A^0 / T_\omega (G\omega) \simeq T_{[\omega]}(A^0 / G) \quad (34) $$

where we have taken $T_\omega (G\omega)$ to be the image of $L_\omega$ under the map

$$ \gamma'_\omega = D^\omega_0 : L_\omega \rightarrow T_\omega A $$

### 7.5 The symplectic structure $\Omega$  

Assume now that $\Sigma$ is an oriented surface, possibly with boundary.

Suppose the Lie algebra $L_\omega$ of $G$ has an $\text{Ad}$-invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$.

If $A, B \in T_\omega A$ then there is a 2-form $\langle A \wedge B \rangle$ on $\Sigma$ whose value on any vectors $v, w \in T_m \Sigma$ is

$$ \langle A \wedge B \rangle(v, w) = \langle A(v), B(w) \rangle - \langle A(w), B(v) \rangle $$

Integrating this 2-form over the compact oriented 2-manifold $\Sigma$ gives

$$ \Omega_\omega(A, B) = \int_\Sigma \langle A \wedge B \rangle \quad (35) $$

This specifies a constant 2-form $\Omega$ (due to $[1]$) on the infinite-dimensional affine space $A$. In fact it is a symplectic structure on $A$.

It is readily verified that $\Omega$ is invariant under the action of $G$.

The space $\overline{A^2}(P; g)$ can be taken to be the dual space $L_\omega^*$. The pairing is given as follows: if $H \in L_\omega$ and $\eta \in \overline{A^2}(P; g)$ then

$$ \langle \eta, H \rangle = \int_\Sigma H\eta $$

Thus the curvature $J(\omega) = \Omega^\omega$ can be viewed as taking values in the dual $L_\omega^*$, and $J$ is then a map

$$ J : A \rightarrow L_\omega^* $$
The adjoint action of $G$ on $LG$ produces a dual action on $LG^*$. With respect to this $J$ is $G$-equivariant.

Suppose now that the compact oriented 2-manifold $\Sigma$ has no boundary, i.e. $\Sigma$ is a closed oriented surface. Then an application of Stokes theorem shows

$$\Omega_\omega(\gamma'_\omega H, B) = \langle J'(\omega)B, H \rangle$$  \hspace{1cm} (36)

Thus $J$ is a moment map for the action of $G$ on the symplectic space $\mathcal{A}$.

This condition (36) implies that, for any flat connection $\omega$, $\Omega$ induces a skew-linear pairing in $H^1(C_\omega) = \ker D^1_\omega / \text{Im} D^0_\omega$:

$$\overline{\Omega}_\omega(A, \overline{B}) = \Omega(A, B)$$

where $A, B \in T_\omega \mathcal{A} = \ker D^1_\omega$, $[\omega]$ is the image of $\omega$ in $\mathcal{A}/G$, $\overline{A} = A + \text{Im} D^0_\omega$, and similarly for $\overline{B}$.

The skew-symmetric pairing $\overline{\Omega}$ may be viewed as as 2-form on the moduli space of flat connections

$$\mathcal{M}^0 = \mathcal{A}^0 / G = J^{-1}(0) / G$$  \hspace{1cm} (37)

The Marsden-Weinstein quotient procedure, for finite-dimensional symplectic manifolds, suggests that $\overline{\Omega}$ is a symplectic structure on (part of) $\mathcal{M}^0$, and this may be rigorously proved with appropriate definitions in the context of the infinite-dimensional space $\mathcal{A}$.

The pairing $\overline{\Omega}$ may be viewed as a "cup-product" on $H^1(C_\omega)$.

### 7.6 The form induced by $\Omega$ for a cell

Now suppose we carry out the construction of $\Omega$ with the disk $D$ (a "cell") as base manifold. Suppose that the boundary $\partial D$ is divided into consecutive arcs $e_1,..., e_N$, with $\partial D = e_N...e_1$, with $e_i$ running from vertex $v_{i-1}$ to vertex $v_i$, for $i = 1,..., N$. Fix a section $s$ over the set $\{v_1,..., v_N\}$. For $\omega \in \mathcal{A}$, the space of all connections on the bundle, let $x_\omega(e_i) \in G$ be specified by

$$\tau_\omega(e_i)v_{i-1} = v_i x_\omega(e_i)$$

where $\tau_\omega(e_i)$ is parallel transport by $\omega$ along $e_i$. Consider the map

$$\Phi : \mathcal{A} \to G^N : \omega \mapsto (x_\omega(e_1),..., x_\omega(e_N))$$
The derivative $d\Phi_\omega : T_\omega A \rightarrow (LG)^N$, defined by

$$d\Phi_\omega(A) = \frac{d}{dt} \bigg|_{t=0} \Phi(\omega + tA)$$

exists, is a linear map, and is given explicitly via the Duhamel formula (38) below.

Let $\Omega'$ be the 2-form on $G^N$ given by

$$\Omega'_x(xA, xB) = \frac{1}{2} \sum_{1 \leq i, j \leq N} \langle f_{i-1}^{-1}A_i, f_{j-1}^{-1}B_j \rangle$$

where $f_i = \text{Ad}(x_{i\ldots x_1})$, and $A = (A_1, \ldots, A_N) \in (LG)^N$ and $B = (B_1, \ldots, B_N) \in (LG)^N$. Compare this with the 2-form $\Omega_\sigma$ defined above in (17).

Let $\omega$ be a flat connection on the bundle over $D$. It is essentially proven in [12] and [8] that for any $A, B \in T_\omega A^0$,

$$\Omega_\omega(A, B) = \Omega'(d\Phi_\omega A, d\Phi_\omega B)$$

8 Relationship between the continuum and discrete cases

Let $\pi : P \rightarrow \Sigma$ be a principal $G$–bundle. We assume that $C$ is a cell–decomposition of $\Sigma$, and each 1–cell $\varepsilon : [0, 1] \rightarrow \Sigma$ is smooth.

Fix a section $\theta$ of $P$ over the 0–cells.

Let $\omega$ be a flat connection on $P$.

Define $x_\omega \in A_C$ as follows: if $\varepsilon$ is a 1–cell running from $a$ to $b$,

$$\tau_\omega(\varepsilon)\theta(a) = \theta(b)x_\omega(\varepsilon)$$

where $\tau_\omega(\varepsilon)$ is parallel–transport by $\omega$ along the oriented 1–cell $\varepsilon$.

Then

$$\tau_\omega(\varepsilon)\theta(b) = \theta(a)x_\omega(\varepsilon)^{-1}$$

So

$$x_\omega(\varepsilon) = x_\omega(\varepsilon)^{-1}$$

Let $\tilde{\varepsilon}_\omega$ be the $\omega$–horizontal lift over $\varepsilon$; thus

$$\tilde{\varepsilon}_\omega(t) = \tau_\omega(\varepsilon|[0, t])\theta(a)$$
Let \( A \in T_\omega \mathbb{A} \). Define
\[
A(e) = -\int_{\hat{e}_\omega} A
\]
It is proven in ... that \( A(e) \) is the derivative of \( \omega \mapsto x\omega(e) \) at \( \omega \) along the vector \( A \):
\[
A(e) = \frac{d}{dt} \bigg|_{t=0} x\omega + tA(e)
\]
(38)
This is the Duhamel formula.
Then
\[
\bar{\tau}^\omega(t) = \tau_\omega([0, t])\theta(b)
\]
\[
= \tau_\omega([0, t])\tau_\omega(e)\theta(a)x\omega(e)^{-1}
\]
\[
= \tau_\omega(e)[0, t]\theta(a)x\omega(e)^{-1}
\]
\[
= \bar{\epsilon}_\omega(1 - t)x\omega(e)^{-1}
\]
So
\[
\bar{\tau}_\omega(1) = \theta(a)x\omega(e)^{-1}
\]
and
\[
A(\bar{\tau}_\omega(t)) = -Ad x\omega(e) A(\bar{\epsilon}_\omega(1 - t))
\]
So
\[
A(\tau) = Ad x\omega(e) \int_{\hat{\epsilon}_\omega} A = -Ad x\omega(e) A(e)
\]
Thus
\[
A \in \mathcal{C}^1_x
\]
Now for \( H \in \mathcal{C}^0_x \),
\[
-\int_{\hat{\epsilon}_\omega} D^\omega H = -\int_{\hat{\epsilon}_\omega} (dH + [\omega, H])
\]
\[
= -\int_{\hat{\epsilon}_\omega} dH
\]
\[
= H(\theta(a)) - Ad x\omega(e)^{-1}H(\theta(b))
\]
Since \( \omega \) is flat, the 2-cell \( \sigma \) has a “horizontal lift” \( \bar{\sigma} \) in the interior \( \sigma^0 \) of \( \sigma \). Then, for any such based oriented 2-cell \( \sigma \), we have
\[
\int_\sigma D^\omega A = \int_\sigma (dA + \omega A)
\]
\[
= \int_\sigma dA
\]
20
\[
= \int_{\partial \sigma} A \\
= A(e_1) + \text{Ad} x_\omega(e_1)^{-1} A(e_2) + \cdots + \text{Ad}(x_\omega(e_{N-1})x_\omega(e_1))^{-1} A(e_N) \\
= D^x_2 A(\sigma)
\]

where \( \partial \sigma = e_N \ldots e_1 \).

Finally, the discussion in section 7.6 shows that the symplectic form \( \Omega \) on the infinite-dimensional space induces, for any flat connection \( \omega \), the 2-form \( \Omega \) on \( \mathfrak{C}_x^1 \).

References


