A QUARTER CENTURY OF WHITE NOISE THEORY

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ABSTRACT. T. Hida created the mathematical theory of white noise in his Carleton Mathematical Lecture Notes “Analysis of Brownian Functionals” (vol. 13, 1975). This theory has since been extensively developed and is now recognized as an important branch of stochastic analysis in the MSC 2000 subject classifications as “white noise theory” with the code number 60H40. We will review the shaping of white noise theory and the progress in various applications during the last quarter century.

1. PRE-THEORY PERIOD (WHITE NOISE WITHOUT A THEORY)

White noise is a sound with equal intensity at all frequencies within a broad band. The sound in a large orchestra before the conductor raises his baton is an example of white noise. Mathematically it is informally defined as a stochastic process \(z(t)\) such that it is independent at different times and has identical distribution with mean 0 and infinite variance in the following sense:

\[
E(z(t)z(s)) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(t-s)x} \cdot 1 \, dx = \delta_0(t - s),
\]

where \(\delta_0\) is the Dirac function at 0. With this definition of white noise \(z(t)\) we can informally derive the following equality for a real-valued function \(f \in L^2(a, b)\),

\[
E\left( \int_a^b f(t)z(t) \, dt \right)^2 = \int_a^b f(t)^2 \, dt.
\]

But what is the definition of the integral \(\int_a^b f(t)z(t) \, dt\)? Moreover, is it possible to define \(z(t)\) for each \(t\)?

We can think of \(z(t)\) as the derivative \(z(t) = \dot{B}(t)\) of a Brownian motion \(B(t)\). Then by regarding \(z(t) \, dt = dB(t)\), the integral \(\int_a^b f(t)z(t) \, dt\) can be defined as the Wiener integral \(\int_a^b f(t) \, dB(t)\). However, since every Brownian path is nowhere differentiable, \(\dot{B}(t)\) does not exist for any \(t\).

More importantly, consider a stochastic process \(f(t)\), is it possible to define the integral \(\int_a^b f(t) \, dB(t)\) or even the integral \(\int_a^b f(t)\dot{B}(t) \, dt\)? It is well-known that the Itô integral \(\int_a^b f(t) \, dB(t)\) is defined for all nonanticipating stochastic processes \(f(t)\) with almost all sample paths in \(L^2(a, b)\). But in order to define the white noise integral \(\int_a^b f(t)\dot{B}(t) \, dt\) we really need to give a rigorous definition of \(\dot{B}(t)\) at least for almost all \(t\).

2. CREATION OF WHITE NOISE THEORY (1975)

The mathematical theory of white noise was created by T. Hida in his 1975 Carleton Mathematical Lecture Notes [123]. In fact, five years earlier he already
envisioned the white noise theory in his Princeton University Press book [116] by discussing Gaussian white noise, Poisson white noise, infinite dimensional rotation groups, etc. But it was in [123] that Hida proposed to use the family \{\hat{B}(t); t \in T\} as a coordinate system to analyze white noise functions. Recently he has advocated calling \{\hat{B}(t); t \in T\} a system of \textit{idealized elemental random variables} as a result of reductionism.

To explain the original idea of Hida in [123], take the Schwartz space \( S(\mathbb{R}) \) of rapidly decreasing real-valued functions on \( \mathbb{R} \). Let \( S'(\mathbb{R}) \) be the dual space of \( S(\mathbb{R}) \). Since \( S(\mathbb{R}) \subset L^2(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) can be identified with its dual space by the Riesz representation theorem, we get a Gel’fand triple

\[
S(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow S'(\mathbb{R}),
\]

By the Minlos theorem there exists a unique measure \( \mu \) on \( S'(\mathbb{R}) \) such that

\[
\int_{S'(\mathbb{R})} e^{i\langle \xi, \cdot \rangle} \ d\mu(x) = e^{-\frac{1}{4}|\xi|^2}, \quad \xi \in S(\mathbb{R}),
\]

where \( |\cdot|_0 \) is the \( L^2(\mathbb{R}) \)-norm. The elements in \( S'(\mathbb{R}) \) can be regarded as \( \hat{B} \) and for this reason the probability space \((S'(\mathbb{R}), \mu)\) is called the \textit{white noise space}.

For simplicity, let \((L^2)^+\) denote the complex Hilbert space \(L^2(S'(\mathbb{R}), \mu)\). By the Wiener-Itô theorem each \( \varphi \in \) has a unique decomposition

\[
\varphi = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L^2_{\text{sym}}(\mathbb{R}^n),
\]

and the \((L^2)^+)\text{-norm of } \varphi \text{ is given by}

\[
||\varphi||_0 = \left( \sum_{n=0}^{\infty} n!|f_n|^2 \right)^{1/2},
\]

where \( I_n \) is the multiple Wiener integral of order \( n \).

Define a space \((L^2)^+\) of test functions and the corresponding space \((L^2)^-\) of generalized functions by

\[
(L^2)^+ = \left\{ \sum_{n=0}^{\infty} I_n(f_n); f_n \in H_{\text{sym}}^{\frac{n+1}{2}}(\mathbb{R}^n) \text{ and } \sum_{n=0}^{\infty} n!|f_n|^2_{H^{-\frac{n+1}{2}}(\mathbb{R}^n)} < \infty \right\},
\]

\[
(L^2)^- = \left\{ \sum_{n=0}^{\infty} I_n(F_n); F_n \in H_{\text{sym}}^{\frac{n+1}{2}}(\mathbb{R}^n) \text{ and } \sum_{n=0}^{\infty} n!|F_n|^2_{H^{-\frac{n+1}{2}}(\mathbb{R}^n)} < \infty \right\},
\]

where \( H^s \) denotes the Sobolev space of order \( s \). Then we have the triple

\[
(L^2)^+ \hookrightarrow (L^2) \hookrightarrow (L^2)^-.
\]

The space \((L^2)^-\) contains elements such as the white noise \( \hat{B}(t) \) as well as the renormalizations \( :\hat{B}(t)^n:\) and \( :e^{\hat{B}(t)}:\) for each fixed \( t \in \mathbb{R} \).

The renormalization of \( \hat{B}(t)^n \) is similar to Itô’s idea of multiple Wiener integral, but the limit is taken in the generalized sense. Consider the case \( n = 2 \). Itô’s idea
is the following limit:

$$B(t)^2 := \lim_{n \to \infty} \sum_{i \neq j} (B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1})) = B(t)^2 - t, \text{ limit in } (L^2)\).$$

On the other hand, here is Hida’s idea to define the renormalization of $\dot{B}(t)^2$ as a generalized function in the space $(L^2)^-$:

$$\left(\frac{B(t + \Delta) - B(t)}{\Delta}\right)^2 := \frac{1}{\Delta^2} \int_t^{t+\Delta} \int_t^{t+\Delta} 1 dB(u)dB(v)$$

$$= \frac{1}{\Delta^2} \lim_{n \to \infty} \sum_{i \neq j} (B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))$$

$$= \frac{1}{\Delta^2} \left[ (B(t + \Delta) - B(t))^2 - \Delta \right].$$

The limit of Equation (2.1), as $\Delta \to 0$, does not exist in $(L^2)$. But if we take the limit in the space $(L^2)^-$ of generalized functions, then the limit, denoted by $\dot{B}(t)^2$, exists and defines a generalized function in $(L^2)^-.$

3. KUBO-TAKENAKA’S CONSTRUCTION (1980)

In 1980 Kubo and Takenaka [260] constructed test and generalized functions on a general space. Let $E' \to E' \to E'$ be a Gel’fand triple. Here $E$ is a real Hilbert space with norm $| \cdot |$ and $E$ is a nuclear space with norms $\{ | \cdot |_p; p \geq 1 \}$ satisfying the conditions:

(a) There exists $0 < \rho < 1$ such that $| \cdot |_1 \leq \rho | \cdot |_1 \leq \rho^2 | \cdot |_2 \leq \cdots \leq \rho^p | \cdot |_p \leq \cdots$

(b) For any $p \geq 1$, there exists $q \geq p$ such that the inclusion map $i_{q,p} : E_q \to E_p$

is a Hilbert-Schmidt operator, where $E_p$ is the completion of $E$ with respect to the norm $| \cdot |_p$.

Let $\mu$ be the probability measure on $E'$ with characteristic function

$$\int_{E'} e^{\langle x, \xi \rangle} d\mu(x) = e^{-\frac{1}{2}|\xi|^2}, \quad \xi \in E.$$

The probability space $(E', \mu)$ is an abstract white noise space. Each $\varphi \in (L^2) \equiv L^2(E', \mu)$ can be uniquely represented by

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x \otimes^n ; f_n \rangle, \quad f_n \in E_{\otimes^n},$$

where $x \otimes^n$ is a Wick tensor (see page 33 in the book [290]. Moreover, the $(L^2)$-norm $\|\varphi\|$ of $\varphi$ is given by

$$\|\varphi\| = \left( \sum_{n=0}^{\infty} n! |f_n|^2 \right)^{1/2},$$

where $| \cdot |$ is the norm on $E_{\otimes^n}$ induced by the norm on $E.$
For $\varphi$ represented by Equation (3.1) and $p \geq 1$, define

$$||\varphi||_p = \left( \sum_{n=0}^{\infty} n! |f_n|^2 \right)^{1/2},$$

where $| \cdot |_p$ is the norm on $\mathcal{E}^\infty_p$ induced by the norm on $\mathcal{E}_p$. Let $(\mathcal{E}_p) = \{ \varphi \in (L^2); ||\varphi||_p < \infty \}$. The projective limit $(\mathcal{E})$ of $\{ (\mathcal{E}_p); p \geq 1 \}$ serves as a space of test functions on the abstract white noise space $(\mathcal{E}', \mu)$. The dual space $(\mathcal{E})^* \equiv (\mathcal{E})$ serves as the corresponding space of generalized functions. Thus we have the Gel'fand triple

$$(\mathcal{E}) \hookrightarrow (L^2) \hookrightarrow (\mathcal{E})^*.$$  

The space $(\mathcal{E})$ of test functions is an infinite dimensional analogue of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ on the finite dimensional space $\mathbb{R}^d$ and has many similar properties as $\mathcal{S}(\mathbb{R}^d)$. For example, it is closed under pointwise multiplication of two test functions. The differential operators, translation operators, scaling operators, Fourier-Gauss transform, Gross--Laplaceian, and number operator are all continuous linear operators on $(\mathcal{E})$. Hence their adjoint operators are continuous on the dual space $(\mathcal{E})^*$. The integral kernel operators can be defined and are continuous linear operators from $(\mathcal{E})$ into $(\mathcal{E})^*$. Moreover, the Hitsuda--Skorokhod integral can be defined as a random variable on the white noise space $\mathcal{S}'(\mathbb{R})$.


In 1991 two important theorems were obtained by Potthoff and Streit [420] (for generalized functions) and by Y.-J. Lee [328] (for test functions). The Potthoff-Streit theorem characterizes generalized functions in the space $(\mathcal{E})^*$ in terms of their $S$-transform

$$S\Phi(\xi) = \langle \Phi, e^{i\cdot\xi} \rangle, \quad \xi \in \mathcal{E}_c,$$

where $\mathcal{E}_c$ is the complexification of $\mathcal{E}$. The theorem says that a complex-valued function $F$ on $\mathcal{E}_c$ is the $S$-transform of a generalized function in $(\mathcal{E})^*$ if and only if it satisfies the following conditions:

(a) For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$;
(b) There exist constants $K, a, p > 0$ such that

$$|F(\xi)| \leq K \exp \left[ a|\xi|^2_p \right], \quad \forall \xi \in \mathcal{E}_c.$$

The Lee theorem describes the space $(\mathcal{E})$ of test functions directly in terms of analyticity and growth condition. It says that a function $\varphi$ on $\mathcal{E}'$ belongs to $(\mathcal{E})$ if and only if for any $p \geq 1$ the function $\varphi$ is analytic on $\mathcal{E}'_{p,c}$ (the complexification of $\mathcal{E}'_p$) and there exists a constant $K_p \geq 0$ such that

$$|\varphi(x)| \leq K_p \exp \left[ \frac{1}{2}|x|^2_{-p} \right], \quad \forall x \in \mathcal{E}'_{p,c}.$$  

On the other hand, test functions in $(\mathcal{E})$ can also be characterized in terms of their $S$-transform [304]. A complex-valued function $F$ on $\mathcal{E}_c$ is the $S$-transform of a test function in $(\mathcal{E})$ if and only if it satisfies the following conditions:

(a) For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$;
(b) For any $a, p > 0$, there exists a constant $K > 0$ such that

$$|F(\xi)| \leq K \exp \left[ a|\xi|^2_{-p} \right], \quad \forall \xi \in \mathcal{E}_c.$$

In 1992 Kondratiev and Streit introduced a family of Gel’fand triples [243] [244]. Let $0 \leq \beta < 1$. For $\varphi$ represented by Equation (3.1) and $p \geq 1$, define

$$\|\varphi\|_{p,\beta} = \left( \sum_{n=0}^{\infty} (n!)^{1+\beta} |f_n|^2_p \right)^{1/2}.$$ 

Let $(E_p)^{\beta} = \{ \varphi \in (L^2); \|\varphi\|_{p,\beta} < \infty \}$. The projective limit $(E)^{\beta}$ of $\{(E_p)^{\beta}; p \geq 1\}$ and its dual space $(E)^{\beta}$ give a new Gel’fand triple

$$(E)^{\beta} \hookrightarrow (L^2) \hookrightarrow (E)^{\beta}.$$ 

Kondratiev and Streit showed in [243] [244] that generalized functions in $(E)^{\beta}$ and test functions in $(E)$ can be characterized as follows. A complex-valued function $F$ on $E_c$ is the $S$-transform of a generalized function in $(E)^{\beta}$ if and only if it satisfies the conditions:

(a) For any $\xi, \eta \in E_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$;

(b) There exist constants $K, a, p > 0$ such that

$$|F(\xi)| \leq K \exp \left[ a|\xi|^2_p \right], \quad \forall \xi \in E_c.$$ 

(5.1)

On the other hand, a complex-valued function $F$ on $E_c$ is the $S$-transform of a test function in $(E)^{\beta}$ if and only if it satisfies the conditions:

(a) For any $\xi, \eta \in E_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$;

(b) For any $a, p > 0$, there exists a constant $K > 0$ such that

$$|F(\xi)| \leq K \exp \left[ a|\xi|^2_p \right], \quad \forall \xi \in E_c.$$ 

(5.2)

Test functions in the space $(E)^{\beta}$ have also been described in [290] in the same spirit of Y.-J. Lee, namely, a function $\varphi$ on $E'$ belongs to $(E)^{\beta}$ if and only if for any $p \geq 1$ the function $\varphi$ is analytic on $E'_{p,c}$ and there exists a constant $K_p \geq 0$ such that

$$|\varphi(x)| \leq K_p \exp \left[ \frac{1}{2} (1 + \beta)|x|^2_p \right], \quad \forall x \in E'_{p,c}.$$ 

The collection $(E)^{\beta}$, $0 \leq \beta < 1$, is an increasing family of generalized functions. Kondratiev and Streit [244] also constructed a space $(E)^{-1}$ of generalized functions dealing with the case $\beta = 1$. But the corresponding space of test functions is not a nuclear space.

Between the union $\cup_{0 \leq \beta < 1} (E)^{\beta}$ and the space $(E)^{-1}$ there is a huge gap of generalized functions. This gap is filled up by the construction of Cochran et al. [82] associated with a sequence of positive real numbers $\{\alpha(n)\}_{n=0}^{\infty}$ satisfying the conditions:

1. $\alpha(0) = 1, \inf_{n\geq0} \alpha(n)\sigma^n > 0$ for some constant $\sigma \geq 1$.

2. $\lim_{n\to\infty} \left( \frac{\alpha(n)}{n!} \right)^{1/n} = 0$.

For $\varphi$ represented by Equation (3.1) and $p \geq 1$, define

$$\|\varphi\|_{p,\alpha} = \left( \sum_{n=0}^{\infty} n! \alpha(n) |f_n|^2_p \right)^{1/2}.$$
Let $[\mathcal{E}_{p, 1}] = \{ \varphi \in (L^2); \|\varphi\|_{p, 1} < \infty \}$ and define $[\mathcal{E}]_1$ to be the projective limit of $\{[\mathcal{E}_{p, 1}]; p \geq 1\}$. The dual space $[\mathcal{E}]_1^*$ is a new space of generalized functions and we have the Gel’fand triple

$$[\mathcal{E}]_1 \hookrightarrow (L^2) \hookrightarrow [\mathcal{E}]_1^*.$$ 

Important examples of $\{\alpha(n)\}$ given by the Bell numbers [82]. A characterization theorem for generalized functions in the space $[\mathcal{E}]_1^*$ is given in [82] with the growth condition: there exist constants $K, a, p > 0$ such that

$$|F(\xi)| \leq KG_\alpha(a|\xi|^2_p)^{1/2},$$

where $G_\alpha(r) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} r^n$. The corresponding characterization theorem for test functions in the space $[\mathcal{E}]_1$ is given in [36] with the growth condition: for any $a, p > 0$, there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq KG_{1/\alpha}(a|\xi|^2_p)^{1/2},$$

where $G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{(\alpha(n))^{1/2}} n^r r^n$.

For the special case $\alpha(n) = (n!)^3$, the growth functions in Equations (5.3) and (5.4) are different from those in Equations (5.1) and (5.2), respectively. Moreover, for a given sequence $\{\alpha(n)\}$, it is impossible in general to sum up $G_\alpha$ and $G_{1/\alpha}$ in close forms. This is the motivation for Asai et al. [33] [35] [38] [39] to introduce CKS-space associated with a growth function.

Let $u$ be a continuous function on $[0, \infty)$ satisfying the following conditions:

1. $\lim_{r \to \infty} \frac{\log u(r)}{r} = \infty$.
2. $\lim \sup_{r \to \infty} \frac{\log u(r)}{r} < \infty$.
3. $u(0) = 1$ and $u$ is increasing.
4. $\log u(x^2)$ is convex for $x \in [0, \infty)$.

Define the Legendre transform $\ell_u$ of $u$ by

$$\ell_u(n) = \inf_{r > 0} \frac{u(r)}{r^n}, \quad n = 0, 1, \ldots,$$

and the dual Legendre transform $u^*$ of $u$ by

$$u^*(r) = \sup_{s > 0} e^{2\sqrt{s r}} \frac{u^2(s)}{s^2}, \quad r \geq 0.$$

For such a function $u$, define a sequence $\alpha_u(n)$ by

$$\alpha_u(n) = \frac{1}{\ell_u(n)n!}, \quad n = 0, 1, \ldots.$$

With this sequence we get a CKS-space denoted by

$$[\mathcal{E}]_u \hookrightarrow (L^2) \hookrightarrow [\mathcal{E}]_u^*.$$ 

Characterization theorems for generalized functions in $[\mathcal{E}]_u^*$ and test functions in $[\mathcal{E}]_u$ are given in [39]. A complex-valued function $F$ on $\mathcal{E}$ is the $S$-transform of a generalized function in $[\mathcal{E}]_u^*$ if and only if it satisfies the conditions:

(a) For any $\xi, \eta \in \mathcal{E}$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$;

(b) There exist constants $K, a, p > 0$ such that

$$|F(\xi)| \leq Ku^*(a|\xi|^2_p)^{1/2}, \quad \forall \xi \in \mathcal{E}.$$
On the other hand, a complex-valued function $F$ on $\mathcal{E}_c$ is the $S$-transform of a test function in $[\mathcal{E}]_u$ if and only if it satisfies the conditions:

(a) For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$;
(b) For any $a, p > 0$, there exists a constant $K > 0$ such that

$$|F(\xi)| \leq Ku(a|\xi|^2_p)^{1/2}, \quad \forall \xi \in \mathcal{E}_c.$$  

6. Applications of white noise theory

There is a wide range of applications of white noise theory. We mention just a few of them below with very brief comments.

1. Integral kernel operators

In 1992 Hida, Obata, and Saitô introduced the integral kernel operators [193]. The generalization to operator-valued generalized functions has been done by Obata, Chung, and Ji, among others. These operators are important for applications in quantum probability.

2. Stochastic integration

In 1981 Kubo and Takenaka related the white noise integral $\int_a^b \varphi(t) \, dB(t)$ to the Itô integral $\int_a^b \varphi(t) \, dB(t)$. The integral $\int_a^b \partial_t \varphi(t) \, dt$ turns out to be the same as the integral introduced by Hida in 1972 [206] and by Skorokhod in 1975 [472]. In the paper [247] Kubo introduced the Itô formula for $f(B(t))$ with $f$ being a generalized function. In 1990 Kuo and Potthoff [302] [303] studied anticipating stochastic integrals and stochastic differential equations. For more information, see the book [296]. Recent progress on white noise methods for stochastic integration has been done by Potthoff and Streit and their collaborators.

3. Infinite dimensional harmonic analysis

In 1975 Hida [123] outlined several things about the infinite dimensional rotation group on the white noise space. Through the years the subgroups of this group, in particular the Lévy group, have been analyzed very much by Hida, Obata, and Saitô. The Lévy Laplacian and related semigroups have been studied extensively by Saitô and his collaborators. Recently Lee and Stan [344] obtained the infinite dimensional Heisenberg inequality. The finite dimensional Paley-Wiener theorem has been generalized to the white noise space by Stan in [473].

4. Stochastic partial differential equations

Since 1991 Øksendal and his collaborators have made significant progress using white noise theory to study stochastic partial differential equations. See the book [217]. Recent achievements of white noise methods to study SPDE’s (in particular the Burgers equation) have been made by Øksendal, Potthoff, Streit, and their collaborators [49] [50] [53] [87] [88] [90] [100] [212] [213].

5. Feynman integral

Feynman integral was one of the motivations for Hida to introduce white noise theory in 1975. Streit and his numerous collaborators have made revolutionary work to treat Feynman integrands as generalized functions in various spaces of generalized functions. In the paper [267] it is shown that with a rather general potential function the associated Feynman integrand is a generalized function in some big space of generalized functions. Recently, Asai et al. [17] [18] have shown that for the
Albeverio-Høegh-Krohn and Laplace transform potentials the associated Feynman integrands are generalized functions in the space $[\mathcal{C}]^*_u$ for some growth function $u$.

6. Random fields and stochastic variational calculus

Since 1988 Hida and Si Si have made significant progress for this application of white noise theory. For further information, see the new book by Hida [184].

7. Intersection local times

In 1991 H. Watanabe [500] used white noise theory to study the local time of self-intersections of Brownian motions. Recently Streit and his collaborators have obtained rather interesting results in [101].

8. Infinite dimensional stochastic differential equations

White noise theory can be used to study the laws of solutions of $\mathcal{S}'(\mathbb{R})$-valued stochastic differential equations. In [311] it is shown that under certain conditions the laws of the solution of an $\mathcal{S}'(\mathbb{R})$-valued SDE induce generalized functions in the space $(\mathcal{S})^*$ of generalized functions. Moreover, these generalized functions satisfy an infinite dimensional partial differential equations.

9. Positive generalized functions

Positive generalized functions in white noise theory was first studied by Potthoff [409] in 1987. These functions are associated with Hida measures on a white noise space. Interesting results have been obtained by Yokoi, Y.-J. Lee, Asai-Kubo-Kuo, on various spaces of generalized functions. For Hida measures on the Kondratiev-Streit space, see the book [290].

10. Dirichlet forms

In 1988 Hida, Potthoff, and Streit [195] initiated the application of white noise theory to Dirichlet forms. Later they were joined by Albeverio and Röckner [23] [24] to achieve revolutionary work on the construction of Dirichlet forms for quantum field theory.

11. Quantum probability

Recent development of quantum probability by Accardi and his collaborators (in particular, Lu, Obata, and Volovich) has shown a strong connection between white noise theory and quantum probability. See the IAS Report [16] and the Volterra Center Preprint [17] and the references therein.

7. References on white noise theory

In the references below we have compiled a list of publications on white noise theory since 1975 with a few exceptions, which are listed for their influence on white noise theory. This list is by no means complete, but it gives some ideas on the development of white noise theory since 1975.

In addition, there have been several conference proceedings and special volumes which contain many papers related to white noise theory:


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