DENSITIES OF 4-RANKS OF $K_2(\mathcal{O})$

ROBERT OSBURN

Abstract. In [1], the authors established a method of determining the structure of the 2-Sylow subgroup of the tame kernel $K_2(\mathcal{O})$ for certain quadratic number fields. Specifically, the 4-rank for these fields was characterized in terms of positive definite binary quadratic forms. Numerical calculations led to questions concerning possible density results of the 4-rank of tame kernels. In this paper, we succeed in giving affirmative answers to these questions.

1. INTRODUCTION

Since the 1960’s, relationships between algebraic K-theory and number theory have been intensely studied. For number fields $F$ and their rings of integers $\mathcal{O}_F$, the K-groups $K_0(\mathcal{O}_F)$, $K_1(\mathcal{O}_F)$, $K_2(\mathcal{O}_F)$, ... were a main focus of attention. From [8] we have

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \times C(F)$$

where $C(F)$ is the ideal class group of $F$, and

$$K_1(\mathcal{O}_F) \cong \mathcal{O}_F^*,$$

the group of units of $\mathcal{O}_F$.

What can we say in general about $K_2(\mathcal{O}_F)$? Garland and Quillen in [3] and [11] showed that $K_2(\mathcal{O}_F)$ is finite. A conjecture of Birch and Tate connects the order of $K_2(\mathcal{O}_F)$ and the value of the zeta function of $F$ at $-1$ when $F$ is a totally real field. For abelian number fields, this conjecture has been confirmed up to powers of 2 [7]. In [12] a 2-rank formula for $K_2(\mathcal{O}_F)$ was given by Tate. Some results on the 4-rank of $K_2(\mathcal{O}_F)$ were given in [9], [10], and [13]. To gain further insight on the 4-rank of $K_2(\mathcal{O}_F)$, we consider the following specific families of fields.

In this paper we deal with the 4-rank of the Milnor K-group $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl}), \mathbb{Q}(\sqrt{2pl}), \mathbb{Q}(\sqrt{-pl}), \mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \mod 8$, $l \equiv 1 \mod 8$ with $\left(\frac{l}{p}\right) = 1$. In [1], the authors

show that for the fields \( E = \mathbb{Q}(\sqrt{pl}) \), \( \mathbb{Q}(\sqrt{2pl}) \) and \( F = \mathbb{Q}(-\sqrt{pl}) \), \( \mathbb{Q}(-2pl) \),

\[
4\text{-rank } K_2(O_E) = 1 \text{ or } 2,
\]

\[
4\text{-rank } K_2(O_F) = 0 \text{ or } 1.
\]

Each of the possible values of 4-ranks is then characterized by checking which ones of the quadratic forms \( X^2 + 32Y^2 \), \( X^2 + 2pY^2 \), \( 2X^2 + pY^2 \) represent a certain power of \( l \) over \( \mathbb{Z} \). This approach makes numerical computations accessible. We should note that this approach involves quadratic symbols and determining the matrix rank over \( \mathbb{F}_2 \) of \( 3 \times 3 \) matrices with Hilbert symbols as entries, see [4]. Fix a prime \( p \equiv 7 \mod 8 \) and consider the set

\[
\Omega = \{ l \text{ rational prime} : l \equiv 1 \mod 8 \text{ and } \left( \frac{l}{p} \right) = \left( \frac{p}{l} \right) = 1 \}.
\]

Let

\[
v = 4\text{-rank } K_2(O_{\mathbb{Q}(\sqrt{pl})})
\]

\[
\mu = 4\text{-rank } K_2(O_{\mathbb{Q}(\sqrt{2pl})})
\]

\[
\sigma = 4\text{-rank } K_2(O_{\mathbb{Q}(\sqrt{-pl})})
\]

\[
\tau = 4\text{-rank } K_2(O_{\mathbb{Q}(\sqrt{-2pl})})
\]

and also consider the sets

\[
\Omega_1 = \{ l \in \Omega : v = 1 \}
\]

\[
\Omega_2 = \{ l \in \Omega : v = 2 \}
\]

\[
\Omega_3 = \{ l \in \Omega : \mu = 1 \}
\]

\[
\Omega_4 = \{ l \in \Omega : \mu = 2 \}
\]

\[
\Lambda_1 = \{ l \in \Omega : \sigma = 0 \}
\]

\[
\Lambda_2 = \{ l \in \Omega : \sigma = 1 \}
\]

\[
\Lambda_3 = \{ l \in \Omega : \tau = 0 \}
\]

\[
\Lambda_4 = \{ l \in \Omega : \tau = 1 \}
\]

We have computed the following (see Table 1 in Appendix): For \( p = 7 \), there are 9730 primes \( l \) in \( \Omega \) with \( l \leq 10^5 \). Among them, there are 4866 primes \( (50.01\%) \) in \( \Omega_1 \) and \( \Omega_3 \) and 4864 primes \( (49.99\%) \) in \( \Omega_2 \) and \( \Omega_4 \). Also, there are 4878 primes \( (50.13\%) \) in \( \Lambda_1 \) and \( \Lambda_3 \) and 4852 primes in \( \Lambda_2 \) and \( \Lambda_4 \). The goal of this paper is to prove the following theorem.

**Theorem 1.1.** For the fields \( \mathbb{Q}(\sqrt{pl}) \) and \( \mathbb{Q}(\sqrt{2pl}) \), 4-rank 1 and 2 each appear with natural density \( \frac{1}{2} \) in \( \Omega \). For the fields \( \mathbb{Q}(\sqrt{-pl}) \) and \( \mathbb{Q}(\sqrt{-2pl}) \), 4-rank 0 and 1 each appear with natural density \( \frac{1}{2} \) in \( \Omega \).
Now consider the tuple of 4-ranks \((v, \mu, \sigma, \tau)\). By Corollary (5.6) in [1], there are eight possible tuples of 4-ranks. For \(p = 7\), among the 9730 primes \(l \in \Omega\) with \(l \leq 10^6\), the eight possible tuples are realized by 1215, 1213, 1228, 1210, 1210, 1228, 1225, 1201 primes \(l\) respectively (see Table 4 in Appendix). And, in fact:

**Theorem 1.2.** Each of the eight possible tuples of 4-ranks appear with natural density \(\frac{1}{8}\) in \(\Omega\).

We will use the following definition throughout this paper.

**Definition 1.3.** For primes \(p \equiv 7 \mod 8, l \equiv 1 \mod 8\) with \((\frac{l}{p}) = (\frac{4}{l}) = 1, \mathcal{K} = \mathbb{Q}(\sqrt{-2p})\), and \(h(\mathcal{K})\) the class number of \(\mathcal{K}\), we say:

- \(l\) satisfies \(A^+\) if and only if \(l = x^2 + 32y^2\) for some \(x, y \in \mathbb{Z}\)
- \(l\) satisfies \(A^-\) if and only if \(l \neq x^2 + 32y^2\) for all \(x, y \in \mathbb{Z}\)
- \(l\) satisfies \(< 2, p >\) if and only if \(l^{-\frac{h(\mathcal{K})}{4}} = 2n^2 + pm^2\) for some \(n, m \in \mathbb{Z}\) with \(m \neq 0 \mod l\)
- \(l\) satisfies \(< 1, 2p >\) if and only if \(l^{-\frac{h(\mathcal{K})}{4}} = n^2 + 2pm^2\) for some \(n, m \in \mathbb{Z}\) with \(m \neq 0 \mod l\).

2. Three Extensions

In this section, we consider three degree eight field extensions of \(\mathbb{Q}\). The idea will be to study composites of these fields and relate Artin symbols to 4-ranks. Rational primes which split completely in a degree 64 extension of \(\mathbb{Q}\) will relate to Artin symbols and thus 4-ranks. Therefore calculating the density of these primes will answer density questions involving 4-ranks.

Let \(\mathcal{D}\) be a Galois extension of \(\mathbb{Q}\), and \(G = \text{Gal}(\mathcal{D}/\mathbb{Q})\). Let \(Z(G)\) denote the center of \(G\) and \(Z(G)'\) denote the fixed field of \(Z(G)\). Let \(p\) be a rational prime which is unramified in \(\mathcal{D}\) and \(\beta\) be a prime of \(\mathcal{D}\) containing \(p\). Let \((\frac{p}{\mathcal{D}})\) denote the Artin symbol of \(p\) and \(\{g\}\) the conjugacy class containing one element \(g \in G\). Throughout this paper, we will make repeated use of the following Lemma.

**Lemma 2.1.** \((\frac{p}{\mathcal{D}}) = \{g\} \subseteq Z(G)\) for some \(g \in Z(G)\) if and only if \(p\) splits completely in \(Z(G)^{'}\).

**Proof.** \((\frac{p}{\mathcal{D}}) = \{g\}\) for some \(g \in Z(G)\) if and only if \((\frac{p}{\mathcal{D}}) = g\) if and only if \((\frac{Z(G)^{'}_\beta}{\mathcal{D}}) = (\frac{p}{\mathcal{D}})\big|_{Z(G)^{'}} = g\big|_{Z(G)^{'}} = Id_{Gal(\mathcal{D}/\mathbb{Q})}\) if and only if \(p\) splits completely in \(Z(G)^{'}\). \(\square\)

Thus if we can show that rational primes split completely in the fixed field of the center of a certain Galois group \(G\), then we know the associated Artin symbol is a conjugacy class containing one element.
Hence we may identify the Artin symbol with this one element and consider the symbol to be an automorphism which lies in \( \mathbb{Z}(G) \). Thus determining the order of \( \mathbb{Z}(G) \) gives us the number of possible choices for the Artin symbol.

2.1. First Extension. Consider \( \mathbb{Q}(\sqrt{2}) \) over \( \mathbb{Q} \). Let \( \epsilon = 1 + \sqrt{2} \in (\mathbb{Z}[\sqrt{2}])^* \). Then \( \epsilon \) is a fundamental unit of \( \mathbb{Q}(\sqrt{2}) \) which has norm \(-1\). Let \( \mathcal{F} = \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}) \). Then \( \mathcal{F} \) has degree 4 over \( \mathbb{Q} \). As

\[
5\sqrt{\epsilon} = 2(\sqrt{2} + \sqrt{\epsilon})^3 - (\sqrt{2} + \sqrt{\epsilon})^2 - 9(\sqrt{2} + \sqrt{\epsilon}) - 9
\]

we have \( \mathcal{F} = \mathbb{Q}(\sqrt{2} + \sqrt{\epsilon}) \). One can readily check that \( \mathcal{F} \) is not a splitting field for the minimal polynomial of \( \sqrt{2} + \sqrt{\epsilon} \) over \( \mathbb{Q} \) and so \( \mathcal{F} \) is not normal over \( \mathbb{Q} \). Since \( \epsilon \in \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}) \) has norm -1 and \( \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1}) \) is a quadratic extension of \( \mathcal{F} \), the normal closure of \( \mathcal{F} \) over \( \mathbb{Q} \) is \( \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1}) \). Set

\[
N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1})
\]

Then \( N_1 \) is Galois over \( \mathbb{Q} \) and \([N_1 : \mathbb{Q}] = 8\).

**Proposition 2.2.** \( \text{Gal}(N_1/\mathbb{Q}) \) is the dihedral group of order 8.

**Proof.** Let \( \text{Gal}(N_1/\mathbb{Q}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\} \). Here are the automorphisms of \( \text{Gal}(N_1/\mathbb{Q}) \):

<table>
<thead>
<tr>
<th></th>
<th>( \sqrt{-1} )</th>
<th>( \sqrt{2} )</th>
<th>( \sqrt{\epsilon} )</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha_3 )</td>
<td>+</td>
<td>-</td>
<td>( \sqrt{\epsilon} )</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha_4 )</td>
<td>+</td>
<td>-</td>
<td>( -\sqrt{\epsilon} )</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha_5 )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha_6 )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha_7 )</td>
<td>-</td>
<td>-</td>
<td>( \sqrt{\epsilon} )</td>
<td>4</td>
</tr>
<tr>
<td>( \alpha_8 )</td>
<td>-</td>
<td>-</td>
<td>( -\sqrt{\epsilon} )</td>
<td>4</td>
</tr>
</tbody>
</table>

Here + and - denote for example \( \alpha_2(\sqrt{-1}) = \sqrt{-1} \) and \( \alpha_2(\sqrt{\epsilon}) = -\sqrt{\epsilon} \) respectively. Note that \( \alpha_1 \) is the identity element of \( \text{Gal}(N_1/\mathbb{Q}) \). Now consider the multiplication table for \( \text{Gal}(N_1/\mathbb{Q}) \):
Here for example $\alpha_3\alpha_3 = \alpha_7$. We note that $\text{Gal}(N_1/Q)$ is generated by $\alpha_3$ and $\alpha_7$ as $\alpha_1 = \alpha_3^2, \alpha_2 = \alpha_7^2, \alpha_4 = \alpha_3\alpha_7^2, \alpha_5 = \alpha_7\alpha_3, \alpha_6 = \alpha_3\alpha_7$, and $\alpha_8 = \alpha_7^3$. Also, $\alpha_3$ has order 2 and $\alpha_7$ has order 4. We need only show $\alpha_3\alpha_7 = \alpha_7^{-1}\alpha_3$ and $\alpha_8^k \neq \alpha_1$ for $k = 1, 2, \text{ or } 3$. From the multiplication table, $\alpha_3\alpha_7 = \alpha_6 = \alpha_8\alpha_3 = \alpha_7^{-1}\alpha_3$ and $\alpha_8^2 = \alpha_2, \alpha_8^3 = \alpha_8$. \hfill \Box

Note that $\alpha_1$ and $\alpha_2$ from the proof of Proposition 2.2 commute with every element of $\text{Gal}(N_1/Q)$ and $\text{Gal}(N_1/Q(\sqrt{2}, \sqrt{-1})) = \{\alpha_1, \alpha_2\}$. It follows

**Corollary 2.3.** $Z(\text{Gal}(N_1/Q)) = \text{Gal}(N_1/Q(\sqrt{2}, \sqrt{-1}))$.

**Proposition 2.4.** If $l \in \Omega$, then $l$ is unramified in $N_1$ over $Q$.

**Proof.** For $l \in \Omega$, $l$ is unramified in $Q(\sqrt{2})$ as $(\frac{2}{l}) = 1$ and $l$ is unramified in $Q(\sqrt{-1})$ as $(\frac{-1}{l}) = 1$. Thus for $l \in \Omega$, $l$ is unramified in the composite field $Q(\sqrt{2}, \sqrt{-1})$.

To see that $l$ is unramified in $Q(\sqrt{\varepsilon})$, we compute the discriminant for the minimal polynomial of $\sqrt{\varepsilon}$ over $Q$. The minimal polynomial of $\sqrt{\varepsilon}$ over $Q$ is $x^4 - 2x^2 - 1$ which has roots $\alpha_1 = \sqrt{\varepsilon}, \alpha_2 = -\sqrt{\varepsilon}, \alpha_3 = \sqrt{\varepsilon}$, and $\alpha_4 = -\sqrt{\varepsilon}$ where $\varepsilon = 1 - \sqrt{2}$. Now,

$$D_{Q(\sqrt{\varepsilon})/Q}(\sqrt{\varepsilon}) = \prod_{i>j} (\alpha_i - \alpha_j)^2$$

$$= (4\varepsilon)(4\varepsilon)(2 - 2i)^2(2 + 2i)^2$$

$$= -1024$$

$$= -2^{10}.$$  

As no odd prime divides $D_{Q(\sqrt{\varepsilon})/Q}(\sqrt{\varepsilon})$, no odd prime divides the field discriminant of $Q(\sqrt{\varepsilon})$ over $Q$. Thus every odd rational prime is unramified in $Q(\sqrt{\varepsilon})$. In particular, $l$ is unramified in $Q(\sqrt{\varepsilon})$. So $l$ is unramified in the composite field $N_1 = Q(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1})$. \hfill \Box
As \( l \in \Omega \) is unramified in \( N_1 \) over \( \mathbb{Q} \), the Artin symbol \( \left( \frac{N_1/\mathbb{Q}}{l} \right) \) is defined for primes \( \beta \) of \( \mathcal{O}_{N_1} \) containing \( l \). Let \( \left( \frac{N_1/\mathbb{Q}}{l} \right) \) denote the conjugacy class of \( \left( \frac{N_1/\mathbb{Q}}{l} \right) \) in \( \text{Gal}(N_1/\mathbb{Q}) \). The primes \( l \in \Omega \) split completely in \( \mathbb{Q}(\sqrt{2}, \sqrt{-1}) \) and

\[
Z(\text{Gal}(N_1/\mathbb{Q}))' = \text{Gal}(N_1/\mathbb{Q}(\sqrt{2}, \sqrt{-1}))' = \mathbb{Q}(\sqrt{2}, \sqrt{-1}).
\]

Thus by Lemma 2.1, we have that \( \left( \frac{N_1/\mathbb{Q}}{l} \right) = \{g\} \subset Z(\text{Gal}(N_1/\mathbb{Q})) \) for some \( g \in \text{Gal}(N_1/\mathbb{Q}) \). As \( Z(\text{Gal}(N_1/\mathbb{Q})) \) has order 2, there are two possible choices for \( \left( \frac{N_1/\mathbb{Q}}{l} \right) \). Combining this statement with Addendum (3.4) from [1], we have

**Remark 2.5.**

\[
\left( \frac{N_1/\mathbb{Q}}{l} \right) = \{id\} \iff l \text{ splits completely in } N_1
\]

\[
\iff l \text{ satisfies } <1, 32>.
\]

\[
\left( \frac{N_1/\mathbb{Q}}{l} \right) \neq \{id\} \iff l \text{ does not split completely in } N_1
\]

\[
\iff l \text{ does not satisfy } <1, 32>.
\]

**2.2. Second Extension.** Consider the fixed prime \( p \equiv 7 \mod 8 \). Note \( p \) splits completely in \( \mathcal{L} = \mathbb{Q}(\sqrt{2}) \) over \( \mathbb{Q} \) and so

\[
p\mathcal{O}_\mathcal{L} = \mathfrak{P}\mathfrak{P}'
\]

for some primes \( \mathfrak{P} \neq \mathfrak{P}' \) in \( \mathcal{K} \). The field \( \mathcal{L} \) has narrow class number \( h^+(\mathcal{L}) = 1 \) as \( h(\mathcal{L}) = 1 \) and \( N_{\mathcal{L}/\mathbb{Q}}(\epsilon) = -1 \) where \( \epsilon = 1 + \sqrt{2} \) is a fundamental unit of \( \mathbb{Q}(\sqrt{2}) \), see [5]. From [1],

**Lemma 2.6.** The prime \( \mathfrak{P} \) which occurs in the decomposition of \( p\mathcal{O}_\mathcal{L} \) has a generator \( \pi = a + b\sqrt{2} \in \mathcal{O}_\mathcal{L} \), unique up to multiplication by the square of a unit in \( \mathcal{O}_\mathcal{L}^* \) for which \( N_{\mathcal{L}/\mathbb{Q}}(\pi) = a^2 - 2b^2 = -p \).

The degree 4 extension \( \mathbb{Q}(\sqrt[4]{2}, \sqrt{\pi}) \) over \( \mathbb{Q} \) has normal closure \( \mathbb{Q}(\sqrt[4]{2}, \sqrt{\pi}, \sqrt{-p}) \) as \( N_{\mathcal{L}/\mathbb{Q}}(\pi) = -p \). Set

\[
N_2 = \mathbb{Q}(\sqrt[4]{2}, \sqrt{\pi}, \sqrt{-p}).
\]

Then \( N_2 \) is Galois over \( \mathbb{Q} \) and \( [N_2 : \mathbb{Q}] = 8 \). Such an extension \( N_2 \) exists since the 2-Sylow subgroup of the ideal class group of \( \mathbb{Q}(\sqrt{-2p}) \) is cyclic of order divisible by 4 [2]. Thus the Hilbert class field of \( \mathbb{Q}(\sqrt{-2p}) \) contains a unique unramified cyclic degree 4 extension over \( \mathbb{Q}(\sqrt{-2p}) \). By Lemma 2.3 in [1], \( N_2 \) is the unique unramified cyclic
degree 4 extension over \( \mathbb{Q}(\sqrt{-2p}) \). Also compare [6]. Similar to the
proof of Proposition 2.2, we have

**Proposition 2.7.** \( \text{Gal}(N_2/\mathbb{Q}) \) is the dihedral group of order 8.

The identity automorphism and the automorphism \( \beta \) induced by
sending \( \sqrt{\pi} \to -\sqrt{\pi} \) commutes with every element of \( \text{Gal}(N_2/\mathbb{Q}) \).

Also

\[
\text{Gal}(N_2/\mathbb{Q}(\sqrt{2}, \sqrt{-p})) = \{id, \beta\}.
\]

Thus

**Corollary 2.8.** \( Z(\text{Gal}(N_2/\mathbb{Q})) = \text{Gal}(N_2/\mathbb{Q}(\sqrt{2}, \sqrt{-p})) \).

**Proposition 2.9.** If \( l \in \Omega \), then \( l \) is unramified in \( N_2 \) over \( \mathbb{Q} \).

*Proof.* Since \( p \equiv 7 \mod{8} \), the discriminant of \( \mathbb{Q}(\sqrt{-2p}) \) is \(-8p\). For
\( l \in \Omega \), we have \( \left( \frac{2}{l} \right) = \left( \frac{-1}{l} \right) = \left( \frac{1}{l} \right) = 1 \) and so \( \left( \frac{-2p}{l} \right) = 1 \). Thus \( l \) is
unramified in \( \mathbb{Q}(\sqrt{-2p}) \). Once again by Lemma (2.3) in [1], \( N_2 \) is the
unique unramified cyclic degree 4 extension over \( \mathbb{Q}(\sqrt{-2p}) \). Hence \( l \) is
unramified in \( N_2 = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p}) \) over \( \mathbb{Q} \).

As \( l \in \Omega \) is unramified in \( N_2 \) over \( \mathbb{Q} \), the Artin symbol \( \left( \frac{N_2/\mathbb{Q}}{l} \right) \) is
defined for primes \( \beta \) of \( \mathcal{O}_{N_2} \) containing \( l \). Let \( \left( \frac{N_2/\mathbb{Q}}{l} \right) \) denote the conjugacy class of \( \left( \frac{N_2/\mathbb{Q}}{l} \right) \) in \( \text{Gal}(N_2/\mathbb{Q}) \). The primes \( l \in \Omega \) split completely
in \( \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p}) \) and

\[
Z(\text{Gal}(N_2/\mathbb{Q})) = \text{Gal}(N_2/\mathbb{Q}(\sqrt{2}, \sqrt{-p})) = \mathbb{Q}(\sqrt{2}, \sqrt{-p}).
\]

By Lemma 2.1, we have that \( \left( \frac{N_2/\mathbb{Q}}{l} \right) = \{h\} \subseteq Z(\text{Gal}(N_2/\mathbb{Q})) \) for some
\( h \in \text{Gal}(N_2/\mathbb{Q}) \). As \( Z(\text{Gal}(N_2/\mathbb{Q})) \) has order 2, there are two possible
choices for \( \left( \frac{N_2/\mathbb{Q}}{l} \right) \). Combining this statement and Lemmas (3.3) and
(3.4) from [1], we have

**Remark 2.10.**

\[
\left( \frac{N_2/\mathbb{Q}}{l} \right) = \{id\} \iff l \text{ splits completely in } N_2 \\
\iff l \text{ satisfies } < 1, 2p > .
\]

\[
\left( \frac{N_2/\mathbb{Q}}{l} \right) \neq \{id\} \iff l \text{ does not split completely in } N_2 \\
\iff l \text{ satisfies } < 2, p > .
\]
2.3. **Third Extension.** Recall for $\mathbb{Q}(\zeta_m)$ with $m$ a positive integer and $p$ a rational prime,

\[
p \text{ splits completely in } \mathbb{Q}(\zeta_m) \iff p \equiv 1 \mod m.
\]

For $l \equiv 1 \mod 8$, we have $l \equiv 1, 9 \mod 16$. Thus for $l \in \Omega$,

\[
l \text{ splits completely in } \mathbb{Q}(\zeta_{16}) \iff l \equiv 1 \mod 16
\]

\[
l \text{ does not split completely in } \mathbb{Q}(\zeta_{16}) \iff l \equiv 9 \mod 16.
\]

These statements yield

**Remark 2.11.**

\[
\left( \frac{\mathbb{Q}(\zeta_{16})/\mathbb{Q}}{l} \right) = \{id\} \iff l \text{ splits completely in } \mathbb{Q}(\zeta_{16})
\]

\[
\iff l \equiv 1 \mod 16.
\]

\[
\left( \frac{\mathbb{Q}(\zeta_{16})/\mathbb{Q}}{l} \right) \neq \{id\} \iff l \text{ does not split completely in } \mathbb{Q}(\zeta_{16})
\]

\[
\iff l \equiv 9 \mod 16.
\]

3. **The Composite and Two Theorems**

In this section we consider the composite field $N_1N_2\mathbb{Q}(\zeta_{16})$. As $N_1$ and $N_2$ are normal extensions of $\mathbb{Q}$, the composite field $N_1N_2$ is a normal extension of $\mathbb{Q}$. Set $\mathcal{G} = \mathbb{Q}(\sqrt{2})$. Then $N_1 = \mathcal{G}(\sqrt{-1}, \sqrt{3})$ is a degree 4 extension of $\mathcal{G}$. Similarly $N_2 = \mathcal{G}(\sqrt{2}, \sqrt{-1})$ is a degree 4 extension of $\mathcal{G}$. Thus the composite field $N_1N_2 = \mathcal{G}(\sqrt{2}, \sqrt{-1}, \sqrt{3})$ is a degree 16 extension of $\mathcal{G}$. Since $[\mathcal{G} : \mathbb{Q}] = 2$, $[N_1N_2 : \mathbb{Q}] = 32$.

For the cyclotomic extension $\mathbb{Q}(\zeta_{16})$ over $\mathbb{Q}$ and its subfield $\mathcal{E} = \mathbb{Q}(\zeta_8) = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$, we have $[\mathbb{Q}(\zeta_{16}) : \mathcal{E}] = 2$. But also note that $N_1N_2 = \mathcal{E}(\sqrt{-1}, \sqrt{3})$ and so $[N_1N_2 : \mathcal{E}] = 8$. Set

\[
L = N_1N_2\mathbb{Q}(\zeta_{16}).
\]

Then $L = \mathcal{E}(\sqrt{-1}, \sqrt{3})$ is a degree 16 extension of $\mathcal{E}$. Since $[\mathcal{E} : \mathbb{Q}] = 4$, we obtain $[L : \mathbb{Q}] = 64$. As $N_1$, $N_2$, and $\mathbb{Q}(\zeta_{16})$ are normal extensions of $\mathbb{Q}$, $L$ is a normal extension of $\mathbb{Q}$.

For $l \in \Omega$, $l$ is unramified in $L$ as it is unramified in $N_1$, $N_2$, and $\mathbb{Q}(\zeta_{16})$. The Artin symbol $(\frac{L/\mathbb{Q}}{\beta})$ is now defined for some prime $\beta$ of $\mathcal{O}_L$ containing $l$. Let $(\frac{L/\mathbb{Q}}{\beta})$ denote the conjugacy class of $(\frac{L/\mathbb{Q}}{\beta})$ in $\text{Gal}(L/\mathbb{Q})$. Letting $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p}) \subset L$, we prove

**Lemma 3.1.** $Z(\text{Gal}(L/\mathbb{Q})) = \text{Gal}(L/M) \cong C_2 \times C_2 \times C_2$. 

Proof. Note that $\text{Gal}(L/M)$ is determined by where the elements $\sqrt{\epsilon}, \sqrt{\pi}, \sqrt{\zeta_8}$ are sent. We consider the automorphisms induced by $\sigma(\sqrt{\epsilon}) = \pm \sqrt{\epsilon}, \pm \sqrt{\pi}, \sigma(\sqrt{\pi}) = \pm \sqrt{\pi}, \pm \sqrt{\pi}$, and $\sigma(\sqrt{\zeta_8}) = \pm \sqrt{\zeta_8}, \pm \sqrt{\zeta_8}$ for some $\sigma \in \text{Gal}(L/M)$. Using the relations $\sqrt{\epsilon} \cdot \sqrt{\epsilon} = -1$, $\sqrt{\pi} \cdot \sqrt{\pi} = -p$, and $\sqrt{\zeta_8} \cdot \sqrt{\zeta_8} = 1$, we have the following table of automorphisms for $\text{Gal}(L/M)$:

<table>
<thead>
<tr>
<th></th>
<th>$\sqrt{\epsilon}$</th>
<th>$\sqrt{\pi}$</th>
<th>$\sqrt{\zeta_8}$</th>
<th>$\sqrt{\epsilon}$</th>
<th>$\sqrt{\pi}$</th>
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<td>-</td>
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<td>b</td>
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<td>c</td>
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</tbody>
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where $\pm$ denotes $\cdot \rightarrow \pm \cdot$ in that order. Every $\sigma \in \text{Gal}(L/M)$ can be written in terms of these three elements. We see that $a, b$, and $c$ each have order 2 and so $|\text{Gal}(L/M)| = 8$. Thus $\text{Gal}(L/M) \cong \langle a > \times \langle b > \times \langle c > \cong C_2 \times C_2 \times C_2$. Now for the inclusion $\text{Gal}(L/M) \subset Z(\text{Gal}(L/Q))$, we need only show $a, b, c \in Z(\text{Gal}(L/Q))$. Since a fixes $\sqrt{\pi}, \sqrt{\zeta_8}$, we only have to check $\sqrt{\epsilon}$. So for any $\sigma \in \text{Gal}(L/Q)$,

$$ a\sigma(\sqrt{\epsilon}) = \begin{cases} a(\sqrt{\epsilon}) = -\sqrt{\epsilon} \\ a(-\sqrt{\epsilon}) = \sqrt{\epsilon} \\ a(\sqrt{\pi}) = -\sqrt{\pi} \\ a(-\sqrt{\pi}) = \sqrt{\pi} \end{cases} \quad \text{and} \quad \sigma a(\sqrt{\epsilon}) = \sigma(-\sqrt{\epsilon}) = \begin{cases} -\sqrt{\epsilon} \\ \sqrt{\epsilon} \\ -\sqrt{\pi} \\ \sqrt{\pi} \end{cases} $$

Thus $a \in Z(\text{Gal}(L/Q))$. Now $b$ fixes $\sqrt{\epsilon}, \sqrt{\zeta_8}$ and so we only check $\sqrt{\pi}$. Thus,

$$ b\sigma(\sqrt{\pi}) = \begin{cases} b(\sqrt{\pi}) = -\sqrt{\pi} \\ b(-\sqrt{\pi}) = \sqrt{\pi} \\ b(\sqrt{\pi}) = -\sqrt{\pi} \\ b(-\sqrt{\pi}) = \sqrt{\pi} \end{cases} \quad \text{and} \quad \sigma b(\sqrt{\pi}) = \sigma(-\sqrt{\pi}) = \begin{cases} -\sqrt{\pi} \\ \sqrt{\pi} \\ -\sqrt{\pi} \\ \sqrt{\pi} \end{cases} $$

So $b \in Z(\text{Gal}(L/Q))$. Similarly $\sigma(\sqrt{\zeta_8}) = \sigma c(\sqrt{\zeta_8})$ and so $c \in Z(\text{Gal}(L/Q))$. For the inclusion $Z(\text{Gal}(L/Q)) \subset \text{Gal}(L/M)$, the idea is to pick an element $\sigma \in \text{Gal}(L/Q)$ such that $\sigma \notin \text{Gal}(L/M)$ and show that $\sigma \notin Z(\text{Gal}(L/Q))$. There are seven cases:

(i) Suppose $\sigma$ does not fix $\sqrt{2}$ but fixes $-\sqrt{2}$, $-\sqrt{-p}$. Choose $\tau$ which sends $-\sqrt{2} \rightarrow -\sqrt{2}$ and fixes $\sqrt{2}$. Then

$$ \sigma \tau(\sqrt{\pi}) = \sigma(\pm \sqrt{\pi}) = \begin{cases} \sigma(\sqrt{\pi}) = \pm \sqrt{\pi} \\ \sigma(-\sqrt{\pi}) = ± \sqrt{\pi} \end{cases} $$

$$ \tau \sigma(\sqrt{\pi}) = \tau(\pm \sqrt{\pi}) = \begin{cases} \tau(\sqrt{\pi}) = \pm \sqrt{\pi} \\ \tau(-\sqrt{\pi}) = ± \sqrt{\pi} \end{cases} $$
Thus $\sigma \notin Z(\text{Gal}(L/Q))$.
(ii) Suppose $\sigma$ does not fix $\sqrt{-1}$, but fixes $\sqrt{2}$, $\sqrt{-p}$. Choose $\tau$ which sends $\sqrt{2} \to -\sqrt{2}$ and fixes $\sqrt{-1}$. Then

$$\sigma \tau(\sqrt{\epsilon}) = \sigma(\pm \sqrt{\epsilon}) = \begin{cases} 
\sigma(\sqrt{\epsilon}) = \pm \sqrt{\epsilon} \\
\sigma(-\sqrt{\epsilon}) = \mp \sqrt{\epsilon}
\end{cases}$$

$$\tau \sigma(\sqrt{\epsilon}) = \tau(\pm \sqrt{\epsilon}) = \begin{cases} 
\tau(\sqrt{\epsilon}) = \pm \sqrt{\epsilon} \\
\tau(-\sqrt{\epsilon}) = \mp \sqrt{\epsilon}
\end{cases}$$

Thus $\sigma \notin Z(\text{Gal}(L/Q))$.
(iii) Suppose $\sigma$ does not fix $\sqrt{2}$, $\sqrt{-1}$ but fixes $\sqrt{-p}$. Choose $\tau$ which sends $\sqrt{-1} \to -\sqrt{-1}$ and fixes $\sqrt{2}$. Then

$$\sigma \tau(\sqrt{\zeta_8}) = \sigma(\pm \sqrt{\zeta_8}) = \begin{cases} 
\sigma(\sqrt{\zeta_8}) = \pm \sqrt{-\zeta_8} \\
\sigma(-\sqrt{\zeta_8}) = \mp \sqrt{-\zeta_8}
\end{cases}$$

$$\tau \sigma(\sqrt{\zeta_8}) = \tau(\pm \sqrt{-\zeta_8}) = \begin{cases} 
\tau(\sqrt{-\zeta_8}) = \pm \sqrt{-\zeta_8} \\
\tau(-\sqrt{-\zeta_8}) = \mp \sqrt{-\zeta_8}
\end{cases}$$

Thus $\sigma \notin Z(\text{Gal}(L/Q))$.
(iv) For $\sigma$ not fixing $\sqrt{-p}$ but fixes $\sqrt{2}$, $\sqrt{-1}$, take $\tau = \sigma$ from case (i) and thus $\sigma \tau(\sqrt{\pi}) \neq \tau \sigma(\sqrt{\pi})$.
(v) For $\sigma$ not fixing $\sqrt{2}$, $\sqrt{-p}$, but fixes $\sqrt{-1}$, take $\tau = \sigma$ from case (ii) and thus $\sigma \tau(\sqrt{\epsilon}) \neq \tau \sigma(\sqrt{\epsilon})$.
(vi) For $\sigma$ not fixing $\sqrt{-p}$, $\sqrt{-1}$, but fixes $\sqrt{2}$, take $\tau = \sigma$ from case (iii) and thus $\sigma \tau(\sqrt{\zeta_8}) \neq \tau \sigma(\sqrt{\zeta_8})$.
(vii) For $\sigma$ not fixing $\sqrt{2}$, $\sqrt{-p}$, $\sqrt{-1}$, take $\tau$ as in case (i) and thus $\sigma \tau(\sqrt{\pi}) \neq \tau \sigma(\sqrt{\pi})$.

Therefore $Z(\text{Gal}(L/Q)) = \text{Gal}(L/M)$. \[]

Now for $l \in \Omega$, $l$ splits completely in $Q(\sqrt{-1})$ and $Q(\sqrt{2}, \sqrt{-p})$ and so splits completely in the composite field $M = Q(\sqrt{2}, \sqrt{-1}, \sqrt{-p})$. From Lemma 3.1,

$$Z(\text{Gal}(L/Q))' = \text{Gal}(L/M)' = M = Q(\sqrt{2}, \sqrt{-1}, \sqrt{-p}).$$

So by Lemma 2.1, we have $(L/Q) = \{k\} \subset Z(\text{Gal}(L/Q))$ for some $k \in \text{Gal}(L/Q)$. As $Z(\text{Gal}(L/Q))$ has order 8, there are eight possible choices for $(L/Q)$. Using Remarks 2.5, 2.10, and 2.11, we now make the following one to one correspondences.

**Remark 3.2.** (i) $(L/Q) = \{id\} \iff l$ splits completely in $L$ over

$$\iff \begin{cases} 
l \text{splits completely in } N_1, \\
N_2, \text{ and } Q(\zeta_{16})
\end{cases} \iff \begin{cases} 
l \text{satisfies } < 1, 32 > \\
l \text{satisfies } < 1, 2p > \\
l \equiv 1 \text{ mod } 16
\end{cases}. $$
(ii) \( \left( \frac{L/Q}{l} \right) \neq \{id\} \iff l \) does not split completely in \( L \). Now there are seven cases.

1. \( \{ l \text{ splits completely in } N_1, \text{ but does not in } N_2 \text{ or } \mathbb{Q}(\zeta_{16}) \} \iff \begin{cases} l \text{ satisfies } < 1, 32 > \\ l \equiv 9 \text{ mod } 16 \end{cases} \)

2. \( \{ l \text{ splits completely in } N_1 \text{ and } N_2, \text{ but does not in } \mathbb{Q}(\zeta_{16}) \} \iff \begin{cases} l \text{ satisfies } < 1, 32 > \\ l \equiv 9 \text{ mod } 16 \end{cases} \)

3. \( \{ l \text{ does not split completely in } N_2, \text{ but does not in } N_1 \text{ or } \mathbb{Q}(\zeta_{16}) \} \iff \begin{cases} l \text{ does not satisfy } < 1, 32 > \\ l \equiv 9 \text{ mod } 16 \end{cases} \)

4. \( \{ l \text{ does not split completely in } N_2 \text{ and } \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_1 \} \iff \begin{cases} l \text{ does not satisfy } < 1, 32 > \\ l \equiv 1 \text{ mod } 16 \end{cases} \)

5. \( \{ l \text{ splits completely in } N_1 \text{ and } \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_2 \} \iff \begin{cases} l \text{ satisfies } < 1, 32 > \\ l \equiv 1 \text{ mod } 16 \end{cases} \)

6. \( \{ l \text{ does not satisfy } < 1, 32 > \\ l \equiv 1 \text{ mod } 16 \end{cases} \)

7. \( \{ l \text{ does not split completely in } N_1, N_2, \text{ or } \mathbb{Q}(\zeta_{16}) \} \iff \begin{cases} l \text{ does not satisfy } < 1, 32 > \\ l \equiv 9 \text{ mod } 16 \end{cases} \).

Now using Theorems (5.2), (5.3), (5.4), and (5.5) from [1], we relate each Artin symbol \( \left( \frac{L/Q}{l} \right) \) to each of the eight possible tuples of 4-ranks.

**Remark 3.3.** From Remark 3.2, case (i) occurs if and only if we have \((2, 2, 1, 1)\). For case (ii),

1. (1) occurs if and only if we have \((1, 2, 0, 1)\)
2. (2) occurs if and only if we have \((2, 1, 1, 0)\)
3. (3) occurs if and only if we have \((2, 1, 0, 1)\)
4. (4) occurs if and only if we have \((2, 2, 0, 0)\)
5. (5) occurs if and only if we have \((1, 1, 0, 0)\)
6. (6) occurs if and only if we have \((1, 1, 1, 1)\)
7. (7) occurs if and only if we have \((1, 2, 1, 0)\).

We can now prove Theorem 1.2.

**Proof.** Consider the set \( X = \{ l \text{ prime : } l \text{ is unramified in } L \text{ and } \left( \frac{L/Q}{l} \right) = C_g \} \) where \( C_g \) is the conjugacy class of \( g \in \text{Gal}(L/Q) \). So we have

\[ X = \{ l \text{ prime : } l \text{ is unramified in } L \text{ and } \left( \frac{L/Q}{l} \right) = \{k\} \subset Z(\text{Gal}(L/Q)) \} \]
for some \( k \in \text{Gal}(L/Q) \). By the \( \check{\text{C}} \)ebotarev Density Theorem, the set \( X \) has natural density \( \frac{1}{64} \) in the set of all primes. Recall

\[
\Omega = \{ l \text{ rational prime} : l \equiv 1 \text{ mod } 8 \text{ and } \left( \frac{l}{p} \right) = \left( \frac{p}{l} \right) = 1 \}
\]

for some fixed prime \( p \equiv 7 \text{ mod } 8 \). By Dirichlet’s Theorem on primes in arithmetic progressions, \( \Omega \) has natural density \( \frac{1}{8} \) in the set of all primes. Thus \( X \) has natural density \( \frac{1}{8} \) in \( \Omega \). By Remark 3.2 and 3.3, each of the eight choices for \( \left( \frac{L/Q}{l} \right) \) is in one to one correspondence with each of the possible tuples of 4-ranks. Thus each of the eight possible tuples of 4-ranks appear with natural density \( \frac{1}{8} \) in \( \Omega \).

\[ \square \]

Now we can prove Theorem 1.1

\textit{Proof.} We see from Remark 3.3, 4-rank \( K_2(\mathcal{O}_{Q(\sqrt{pl})}) = 1 \) in cases (ii), parts (1), (5), (6), and (7), 4-rank \( K_2(\mathcal{O}_{Q(\sqrt{2pl})}) = 2 \) in case (i) and case (ii) parts (1), (4), and (7), 4-rank \( K_2(\mathcal{O}_{Q(\sqrt{-pl})}) = 0 \) in case (ii) parts (1), (3), (4), and (5), 4-rank \( K_2(\mathcal{O}_{Q(\sqrt{-2pl})}) = 1 \) in case (i) and case (ii) parts (1), (3), and (6). As each of the 4-rank tuples occur with natural density \( \frac{1}{8} \), we have for the fields \( Q(\sqrt{pl}) \) and \( Q(\sqrt{2pl}) \), 4-rank 1 and 2 each appear with natural density \( 4 \cdot \frac{1}{8} = \frac{1}{2} \) in \( \Omega \). For the fields \( Q(\sqrt{-pl}) \) and \( Q(\sqrt{-2pl}) \), 4-rank 0 and 1 each appear with natural density \( 4 \cdot \frac{1}{8} = \frac{1}{2} \) in \( \Omega \).

\[ \square \]

\textbf{Appendix}

The following tables motivated possible density results of 4-ranks of tame kernels. We consider primes \( l \in \Omega \) with \( l \leq N \) for a fixed prime \( p \equiv 7 \text{ mod } 8 \) and positive integer \( N \). For Tables 1, 2, and 3, we consider the sets \( \Omega_1, \ldots, \Omega_4 \) and \( \Lambda_1, \ldots, \Lambda_4 \) as in the Introduction. For Tables 4, 5, and 6, we consider the sets

\[ I_1 = \{ l \in \Omega : \text{4-rank tuple is } (1,1,0,0) \} \]
\[ I_2 = \{ l \in \Omega : \text{4-rank tuple is } (1,1,1,1) \} \]
\[ I_3 = \{ l \in \Omega : \text{4-rank tuple is } (2,1,1,0) \} \]
\[ I_4 = \{ l \in \Omega : \text{4-rank tuple is } (2,1,0,1) \} \]
\[ I_5 = \{ l \in \Omega : \text{4-rank tuple is } (1,2,1,0) \} \]
\[ I_6 = \{ l \in \Omega : \text{4-rank tuple is } (1,2,0,1) \} \]
\[ I_7 = \{ l \in \Omega : \text{4-rank tuple is } (2,2,0,0) \} \]
\[ I_8 = \{ l \in \Omega : \text{4-rank tuple is } (2,2,1,1) \} \].
**Densities of 4-Ranks of $K_4(O)$**

**Table 1: $p = 7$**

<table>
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<tr>
<th>Cardinality</th>
<th>$N = 6620$</th>
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<th>$N = 25000$</th>
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<td>160</td>
<td>48.93</td>
<td>4864</td>
<td>49.99</td>
</tr>
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</table>

| $\Lambda$    |            |       |             |       |               |       |
| $\Lambda_1$  | 46         | 48.42 | 167         | 51.07 | 4878          | 50.13 |
| $\Lambda_2$  | 49         | 51.58 | 160         | 48.93 | 4852          | 49.87 |
| $\Lambda_3$  | 53         | 55.79 | 172         | 52.60 | 4878          | 50.13 |
| $\Lambda_4$  | 42         | 44.21 | 155         | 47.40 | 4852          | 49.87 |

**Table 2: $p = 23$**

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<td>4831</td>
<td>49.59</td>
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</table>

| $\Lambda$    |            |       |             |       |               |       |
| $\Lambda_1$  | 54         | 55.10 | 164         | 50.00 | 4912          | 50.42 |
| $\Lambda_2$  | 44         | 44.90 | 164         | 50.00 | 4830          | 49.58 |
| $\Lambda_3$  | 50         | 51.02 | 168         | 51.22 | 4876          | 50.05 |
| $\Lambda_4$  | 48         | 48.98 | 160         | 48.78 | 4866          | 49.95 |

**Table 3: $p = 31$**

<table>
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| $\Lambda$    |            |       |             |       |               |       |
| $\Lambda_1$  | 54         | 55.67 | 168         | 51.69 | 4930          | 50.54 |
| $\Lambda_2$  | 43         | 44.33 | 157         | 48.31 | 4824          | 49.46 |
| $\Lambda_3$  | 47         | 48.45 | 159         | 48.92 | 4943          | 50.68 |
| $\Lambda_4$  | 50         | 51.55 | 166         | 51.08 | 4811          | 49.32 |
TABLE 4: $p = 7$

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TABLE 5: $p = 23$

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TABLE 6: $p = 31$

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<thead>
<tr>
<th>Cardinality</th>
<th>$N = 6620$</th>
<th>%</th>
<th>$N = 25000$</th>
<th>%</th>
<th>$N = 1000000$</th>
<th>%</th>
</tr>
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<tbody>
<tr>
<td>$\Omega$</td>
<td>97</td>
<td>325</td>
<td>9754</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_1$</td>
<td>14</td>
<td>14.43</td>
<td>38</td>
<td>11.69</td>
<td>1246</td>
<td>12.77</td>
</tr>
<tr>
<td>$I_2$</td>
<td>12</td>
<td>12.37</td>
<td>49</td>
<td>15.08</td>
<td>1203</td>
<td>12.33</td>
</tr>
<tr>
<td>$I_3$</td>
<td>11</td>
<td>11.34</td>
<td>38</td>
<td>11.69</td>
<td>1214</td>
<td>12.45</td>
</tr>
<tr>
<td>$I_4$</td>
<td>15</td>
<td>15.47</td>
<td>45</td>
<td>13.85</td>
<td>1188</td>
<td>12.18</td>
</tr>
<tr>
<td>$I_5$</td>
<td>10</td>
<td>10.31</td>
<td>40</td>
<td>12.31</td>
<td>1227</td>
<td>12.58</td>
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<tr>
<td>$I_6$</td>
<td>13</td>
<td>13.40</td>
<td>42</td>
<td>12.92</td>
<td>1240</td>
<td>12.71</td>
</tr>
<tr>
<td>$I_7$</td>
<td>12</td>
<td>12.37</td>
<td>43</td>
<td>13.23</td>
<td>1256</td>
<td>12.88</td>
</tr>
<tr>
<td>$I_8$</td>
<td>10</td>
<td>10.31</td>
<td>30</td>
<td>9.23</td>
<td>1180</td>
<td>12.10</td>
</tr>
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</table>
DENSITIES OF 4-RANKS OF $K_2(\mathcal{O})$

ACKNOWLEDGMENTS

This paper is part of the author’s LSU dissertation written under the direction of J. Hurrelbrink. The author deeply thanks his advisor and P.E. Conner.

REFERENCES


DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803

E-mail address: osburn@math.lsu.edu