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FURTHER RESULTS ON LYAPUNOV FUNCTIONS AND DOMAINS OF ATTRACTION FOR PERTURBED ASYMPTOTICALLY STABLE SYSTEMS

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Abstract. We present new theorems characterizing robust Lyapunov functions and infinite horizon value functions in optimal control as unique viscosity solutions of partial differential equations. We use these results to further extend Zubov's method for representing domains of attraction in terms of partial differential equation solutions.
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1 Introduction

The theories of Lyapunov functions and domains of attraction form the basis for much of current work in stability theory (cf. [2, 6, 8, 18]). An important result in this area is the *Zubov method* (cf. [1, 9, 11, 23, 24]), which gives conditions under which the domain of attraction of an asymptotically stable fixed point of $\dot{x} = f(x)$ is $v^{-1}([0, 1))$, where v is the solution of the *Zubov equation*

$$Dv(x) \cdot f(x) = -H(x)[1 - v(x)]\sqrt{1 + ||f(x)||^2}, \;\; x \in \mathbb{R}^N$$

for suitable functions H. In [5, 6, 8], Zubov's method was extended to the important case of perturbed asymptotically stable systems $\dot{x} = f(x, a)$ for which the fixed point 0 is stable under any perturbation a. These perturbations, taken to be $\mathcal{A} := \{\text{measurable functions}\, \alpha : [0, \infty) \to A\}$ for a given compact set A, are used to represent uncertainties and exogenous effects, rather than controls. The main results in [5, 6, 8] are partial differential equations (PDE) characterizations for *robust domains of attraction* and *robust Lyapunov functions* for perturbed dynamics (cf. §2 below for the relevant definitions). Under the conditions in [6], the robust domain of attraction $\mathcal{D}_o$ for the perturbed system $\dot{x} = f(x, a)$ is $v^{-1}([0, 1))$, where v is the unique bounded viscosity solution of the *generalized Zubov equation*

$$\inf_{a\in A}\{-Dv(x)\cdot f(x,a)-g(x,a)+v(x)g(x,a)\}=0,\quad x\in\mathbb{R}^N \tag{1}$$

that vanishes at the origin, under certain restrictions on g. The solution v of (1) is a robust Lyapunov function for the perturbed dynamics f. Also, if mild assumptions are added on g and f, then v is locally Lipschitz (cf. §4.2.2 below). The results from [6] form the basis for discrete approximations of Lyapunov functions and $\mathcal{D}_o$ (cf. [5]). In [5, 6, 8], the function g in (1) is assumed to satisfy[1]

$$\begin{array}{ll} (i) & \inf\{g(x,a): x\notin B_r, a\in A\}\ \geq\ g_o\ >\ 0 \\ (ii) & g(x,a)>0 \quad \forall\, x\neq 0, \forall\, a\in A \end{array} \tag{2}$$

for some constants g_o and r. However, one can easily find equations (1) which admit several bounded solutions which are null at the origin when (2) is not satisfied. Here is an example where this occurs:

Example 1.1 Define the functions $f,g:\mathbb{R}\times[-1,1]\to\mathbb{R}$ by

$$f(x,a):=\begin{cases} 1-\frac{a}{x}, & \text{if } x\leq -1,\\ -x+ax^2, & \text{if } -1\leq x\leq 1,\\ \frac{a}{x}-1, & \text{if } x\geq 1\end{cases}$$

and

$$g(x,a)\equiv\gamma(x):=\begin{cases} |\sin(\pi x)|, & -1\leq x\leq 1\\ 0, & \text{otherwise}\end{cases}.$$

Then the robust domain of attraction $\mathcal{D}_o$ is $(-1,+1)$. Let $\mathcal{M}$ denote the set of all measurable functions $\alpha:[0,\infty)\to A:=[-1,+1]$ and $\mathrm{Traj}_x(f)$ denote the set of all solutions $\phi:[0,\infty)\to\mathbb{R}$ of $\dot{y}=f(y,a)$ starting at x for $a\in\mathcal{M}$. One bounded solution[2] of the corresponding generalized Zubov equation

$$\inf_{a\in A}\{-f(x,a)\cdot Dv(x)\}+(v(x)-1)\gamma(x)\ =\ 0 \tag{3}$$

on $\mathbb{R}$ is $v_1=1-\exp(-W_1)$, where W_1 is the maximal cost function

$$W_1(x)=\sup\left\{\int_0^\infty \gamma(\phi(t))\,\mathrm{d}t:\phi\in\mathrm{Traj}_x(f)\right\}$$

This follows from an elementary consideration of semidifferentials.[3] Notice that $0\leq v_1\leq 1$ on $\mathbb{R}$ and $v_1(0)=0$. Other bounded solutions of (3) on $\mathbb{R}$ are

$$v_2(x)=\begin{cases} v_1(x), & x>-1\\ k, & x\leq -1\end{cases},\quad k>1 \text{ constant}$$

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[1] $B_r:=\{x\in\mathbb{R}^N:||x||\leq r\}$ for all $r>0$, where $||x||=(x_1^2+x_2^2+\ldots+x_N^2)^{1/2}$.

[2] We will understand our Zubov equation solutions to be in the discontinuous viscosity sense (cf. §2, Definition 2.2).

[3] By the arguments of [6], v_1 is a solution of (3) on any interval of the form $(-\delta,+\delta)$ for $\delta\in(0,1)$, since on this interval, v_1 is the Kružkov transformation of a maximal cost value function for a cost function satisfying (2). Evidently v_1 is a solution of (3) outside

As before, $v_2(0) = 0$, but $v_2(-1) > v_1(-1)$. Notice that v_1 and v_2 are continuous at the origin.

Remark 1.2 Our hypotheses will imply $v^{-1}([0,1)) = \mathcal{D}_o$, where v is the unique bounded solution of the corresponding generalized Zubov equation that satisfies $v(0) = 0$ and that is continuous at the origin. One hypothesis we will make will be g quasi-stability of $\mathbb{R}^N$, which is roughly the condition that trajectories on $[0, \infty)$ with finite total cost must approach the origin (cf. §2 for the definition of g quasi-stability). In the previous example, this condition is not satisfied, since $\phi(t) \equiv 1$ gives 0 total costs. On the other hand, all the other hypotheses of this note are satisfied in this example, which shows that our quasi-stability hypothesis cannot be dropped.

Example 1.1 motivates our study of the solutions of the generalized Zubov equation (1), and the sublevel sets of these solutions, for general costs g. This note will develop *uniqueness* theory for solutions of (1), which includes the uniqueness characterizations given in [6], and also applies to cases which are not tractable by the known results, e.g., cases where g is non-Lipschitz or violates (2). We also study the *regularity* of Zubov equation solutions, and the sublevel sets of these solutions, under relaxed assumptions on f and g. Our results have the following novel features:

1. Our results are based on extensions of results in [13, 14, 15, 16, 17, 21] on uniqueness of solutions for the infinite horizon Hamilton-Jacobi equation, namely, Theorem 1 and Propositions 3.5 and 3.6 in §3. The infinite horizon equation is the same as the exit time equation. Whereas [15, 16, 17] assume that the undiscounted infinite horizon Lagrangian is nonnegative, our results apply for Lagrangians which could be null or negative. It is natural to consider optimization with Lagrangians that take both positive and negative values, to allow cost minimization in one part of the state space and maximization elsewhere. Results on undiscounted exit problems with negative Lagrangians were given in [14], which requires controllability to the so-called positivity set of the Lagrangian. This controllability condition is not needed below, nor do we need the uniform positive lower bounds on the interest rates used in [22]. Theorem 1 does not put any growth or lower bound assumptions on solutions, nor does it require any controllability at the origin. On the other hand, the earlier uniqueness characterizations

$[-1, +1]$ where it is constant. It remains to check the semidifferential condition for v_1 to be a solution of (3) at ± 1 (cf. [3]). To check this condition at -1, it suffices to check that (i) $-p - |p| \leq 0$ for all $p \in D^+(v_1)^\star(-1)$ and (ii) $-p - |p| \geq 0$ for all $p \in D^-(v_1)_\star(-1)$. Condition (i) is trivial, while (ii) follows because if $p \in D^-(v_1)_\star(-1)$, then

$$-p \geq \liminf_{h \downarrow 0} \frac{(v_1)_\star(-1+h) - (v_1)_\star(-1) - ph}{h} \geq 0,$$

since $(v_1)_\star(-1+h) \leq (v_1)_\star(-1)$ for $h > 0$ near 0. A similar argument applies at $+1$. This argument also shows that the functions v_2 we are about to give are solutions of (3) on $\mathbb{R}$.

for Hamilton-Jacobi equations (cf. [3, 14, 15, 16, 17, 21, 22]) prove uniqueness of solutions in classes of functions which are either proper or bounded below by a finite constant, or which satisfy an asymptotic condition at the boundary of the domain. Therefore, in addition to applying to problems with more general Lagrangians, our results extend previous work by allowing more general comparison functions, including functions which are negative and neither bounded-from-above nor bounded-from-below.

2. Our uniqueness theory for (1) applies for general functions g, and gives stronger conclusions than the uniqueness theory of [5, 6, 8]. Clearly, a function w is a solution of (1) exactly when $-w$ is a solution of the usual infinite time Hamilton-Jacobi equation (8) with interest rate $h = g$ and Lagrangian $\ell = -g$. However, since the usual uniqueness results for (8) require nonnegative ℓ or strictly positive h (cf. [3, 21, 22]), these results cannot in general be applied to (1) when g is nonnegative. Moreover, the known results on (1) (cf. [5, 6, 8]) require condition (2), and a growth condition such as Lipschitzness for g, and therefore cannot be applied to general situations. Since we allow general possibly non-Lipschitz costs g, including cases where (2) is not satisfied (cf. Remark 4.9 and §5), our theory can be regarded as an extension of the results of [13] on optimal control for non-Lipschitz dynamics. By allowing degenerate costs g, we obtain solutions of (1) with properties not found in the Zubov equation solutions of [6] (cf. §4.2.2 and Remark 4.8). The unique solutions of (1) are robust Lyapunov functions for f, and are the Kružkov transformations of maximal cost type robust Lyapunov functions V_L for f. The functions V_L are in turn unique solutions of

$$\inf_{a \in A} \{-f(x,a) \cdot Dv(x) - g(x,a)\} = 0 \tag{4}$$

on $\mathcal{D}_o$. This generalizes the PDE characterizations for (1) and (4) in [6]. On the other hand, the uniqueness characterizations for (1) and (4) in [6] all follow from the results given below. Therefore, we obtain new classes of 'flat' maximal cost type robust Lyapunov functions, corresponding to degenerate cost functions g, which can be characterized as unique PDE solutions (cf. Remark 4.8). This leads to new characterizations of $\mathcal{D}_o$ as sublevel sets of value functions for degenerate instantaneous costs (cf. Corollary 4.6).

This note is organized as follows. In §2, we give definitions and lemmas from [3, 6]. In §3, we give general uniqueness theory for solutions of infinite horizon Hamilton-Jacobi equations. In §4, we apply these results to stabilization, and investigate the regularity and uniqueness of solutions of (1) and (4). These solutions are shown to be Lyapunov functions with special properties not found in the Zubov equation solutions in [6] (cf. §4.2.2). Our results are based on recent variations of the Filippov-Ważewski Relaxation Theorem, which we review in Appendix A. In §5, we illustrate our results with an example from [6]. We close in §6 by discussing extensions and directions for future research.

2 Definitions and Lemmas

This note is concerned with perturbed systems of the form $\dot{x} = f(x,a)$, $x(0) = x_o$ where the input a represents exogenous effects and uncertainties in the design of the control system. As in [6], we will assume the following:

(A_1) A is a nonempty compact metric space.

(A_2) $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ is bounded and continuous, where N is a fixed positive integer. Also, $f(x,a)$ is uniformly locally Lipschitz, meaning, for each $R > 0$, there is a constant $L_R > 0$ such that if $||x||, ||y|| \leq R$ and $a \in A$, then $||f(x,a) - f(y,a)|| \leq L_R||x - y||$.

(A_3) $f(0,a) = 0$ for all $a \in A$.

In §4, we use functions g as the cost functions for *greatest cost* Lyapunov functions, and we use cost functions ℓ, which we will refer to as **Lagrangians**, when we study *least cost* optimal control value functions in §3. We always assume that g and ℓ satisfy the following:

(A_4) $g, \ell : \mathbb{R}^N \times A \to \mathbb{R}$ are continuous, $\ell(0,a) = g(0,a) = 0$ for all $a \in A$, and g is nonnegative.

In our main applications, we take $\ell(x,a) \leq 0$ for all (x,a), but our theory applies to general continuous ℓ (cf. §3.2). Assumptions (A_1)-(A_3) imply that for each $a \in \mathcal{A} := \{$ measurable functions $\alpha : [0,\infty) \to A\}$ and $x_o \in \mathbb{R}^N$, the system

$$\dot{x}(t) = f(x(t), a(t)), \quad x(0) = x_o \tag{5}$$

has a unique (classical) solution defined on $[0,\infty)$, which will be denoted by $\phi(\cdot, x_o, a)$, and called the **trajectory** of f for a starting at x_o. Elements of $\mathcal{A}$ are called **controls** or **inputs**. For any f and A satisfying (A_1)-(A_3), $\mathrm{Traj}_{x_o}(f) := \{\phi(\cdot, x_o, a) : a \in \mathcal{A}\}$.

We sometimes also use the **relaxed controls** $\mathcal{A}^r$, which is the set of all measurable functions $\alpha : [0,\infty) \to A^r$, where A^r is the set of all Radon probability measures on A (cf. [3]). Recall (cf. [3]) that A^r is a compact metric space. By $\mathcal{A}^r \ni \alpha_n \to \alpha \in \mathcal{A}^r$ weak-$\star$, we mean that for all $t \geq 0$ and all Lebesgue integrable functions $B : [0,t] \to C(A)$,

$$\lim_{n\to\infty} \int_0^t \int_A (B(s))(a)\,\mathrm{d}(\alpha_n(s))(a)\,\mathrm{d}s = \int_0^t \int_A (B(s))(a)\,\mathrm{d}(\alpha(s))(a)\,\mathrm{d}s \tag{6}$$

where $C(A)$ is the set of all real-valued continuous functions on A. For any $M \in \mathbb{N}$ and any function $\Phi : \mathbb{R}^N \times A \to \mathbb{R}^M$, we define $\Phi^r : \mathbb{R}^N \times A^r \to \mathbb{R}^M$

by $\Phi^r(x,m) := \int_A \Phi(x,a)\,\mathrm{d}m(a)$. Notice that (A_1)-(A_3) hold with f replaced by f^r and A replaced by A^r. We also use $\phi(\cdot,x,\alpha)$ to denote the solution of $\dot{y} = f^r(y,\alpha)$ defined on $[0,+\infty)$ starting at x for each $\alpha \in \mathcal{A}^r$. Therefore, $\mathrm{Traj}_x(f^r) = \{\phi(\cdot,x,\alpha) : \alpha \in \mathcal{A}^r\}$. This extends our original definition of ϕ, since we can view $\mathcal{A}$ as the subset of $\mathcal{A}^r$ consisting of all Dirac probability measure valued relaxed controls. When $\alpha \in \mathcal{A}^r$, we call $\phi(\cdot,x,\alpha)$ the **relaxed trajectory** of f for α starting at x. Recall the following Compactness Lemma on relaxed controls (cf. [3]):

Lemma 2.1 Let (A_1)-(A_2) hold, let $\{\alpha_n\}$ be a sequence in $\mathcal{A}^r$, and let $c > 0$. Then there exists a subsequence of $\{\alpha_n\}$ (which we do not relabel) and an $\alpha \in \mathcal{A}^r$ such that (i) $\alpha_n \to \alpha$ weak-$\star$ on $[0,c]$ and such that (ii) if $x_n \to x$ in $\mathbb{R}^N$, then $\phi(\cdot,x_n,\alpha_n) \to \phi(\cdot,x,\alpha)$ uniformly on $[0,c]$.

We sometimes also use a function h, representing a discount rate, which we always assume satisfies

(A_5) $\quad h : \mathbb{R}^N \times A \to [0,+\infty)$ is continuous.

We say that f is **uniformly locally asymptotically stable (ULAS)** provided that

($\star\star$) There are $\beta_f \in \mathcal{KL}$ and $r > 0$ such that $||\phi(t,x,\alpha)|| \le \beta_f(||x||,t)$ for all $x \in B_r$, $t \ge 0$, and $\alpha \in \mathcal{A}$.

and that f is **uniformly locally exponentially stable (ULES)** if ($\star\star$) holds for $\beta_f(s,t) = Cse^{-\sigma t}$ for some positive constants C and σ.[4] When f is ULAS, we set $t(x,\alpha) := \inf\{t \ge 0 : ||\phi(t,x,\alpha)|| \le r\}$ for all $(x,\alpha) \in \mathbb{R}^N \times \mathcal{A}^r$, where $\inf \emptyset$ is defined to be $+\infty$. Also,

$$\mathcal{D} := \left\{ x \in \mathbb{R}^N : \lim_{t\to+\infty} \phi(t) = 0 \quad \forall \phi \in \mathrm{Traj}_x(f) \right\}$$

and

$$\mathcal{D}_o := \left\{ x \in \mathbb{R}^N : \sup_{\alpha\in\mathcal{A}} t(x,\alpha) < +\infty \right\}.$$

We call $\mathcal{D}$ the **domain of attraction** of f, and $\mathcal{D}_o$ is called the **robust domain of attraction** of f. As shown in [6], the sets $\mathcal{D}$ and $\mathcal{D}_o$ may differ for ULES dynamics satisfying (A_1)-(A_3). Since relaxed trajectories of f can be uniformly approximated by trajectories of f on compact intervals without changing the initial value, one shows that if f is ULAS, then so is f^r. For each

[4] Recall (cf. [19]) that $\mathcal{K}^\infty$ is defined to be the set of all strictly increasing functions $F : [0,\infty) \to [0,\infty)$ which satisfy (i) $F(0) = 0$ and (ii) $F(x) \to +\infty$ as $x \to +\infty$. Also, $\mathcal{KL}$ is the set of all continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ for which (i) $\beta(\cdot,t) \in \mathcal{K}^\infty$ for all $t \ge 0$, (ii) $\beta(s,\cdot)$ is decreasing for all $s \ge 0$, and (iii) $\beta(s,t) \to 0$ as $t \to +\infty$ for all $s \ge 0$.

open set $\mathcal{G} \subseteq \mathbb{R}^N$, let $C^1(\mathcal{G})$ denote the set of all continuously differentiable functions $F : \mathcal{G} \to \mathbb{R}$. For each $S \subseteq \mathbb{R}^N$, we set

$$w_\star(x) = \liminf_{S \ni y \to x} w(y) \quad \text{and} \quad w^\star(x) = \limsup_{S \ni y \to x} w(y)$$

for all $x \in S$ and locally bounded functions $w : S \to \mathbb{R}$. Then $w_\star$ is lower semicontinuous and $w^\star$ is upper semicontinuous. Also, $w_\star = w^\star = w$ at all points of continuity of w. We will find conditions under which the **generalized Zubov equation**

$$\inf_{a \in A}\{-Dv(x) \cdot f(x,a) - g(x,a) + v(x)g(x,a)\} = 0 \tag{7}$$

has a unique bounded continuous solution v on $\mathbb{R}^N$ that satisfies the two conditions $v(0) = 0$ and $v^{-1}([0,1)) = \mathcal{D}_o$. Also, we consider solutions of

$$\sup_{a \in A}\{-f(x,a) \cdot Dv(x) - \ell(x,a) + h(x,a)v(x)\} = 0 \tag{8}$$

which we refer to as the **infinite horizon Hamilton-Jacobi equation** (for the dynamics f, Lagrangian ℓ, and interest rate h). Since solutions of (7)-(8) may not be differentiable or even continuous (cf. [3]), we consider solutions of these equations in the viscosity sense, by which we mean the following:

Definition 2.2 Let $\mathcal{G} \subseteq \mathbb{R}^N$ be open, $S \supseteq \mathcal{G}$, $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous, and $w : S \to \mathbb{R}$ be locally bounded. We call w a **(discontinuous viscosity) solution** of $F(x, w(x), Dw(x)) = 0$ on $\mathcal{G}$ provided the following two conditions hold:

- (C_1) If $\gamma \in C^1(\mathcal{G})$ and $x_o \in \mathcal{G}$ is a local minimum of $w_\star - \gamma$, then $F(x_o, w_\star(x_o), D\gamma(x_o)) \geq 0$.
- (C_2) If $\lambda \in C^1(\mathcal{G})$ and $x_1 \in \mathcal{G}$ is a local maximum of $w^\star - \lambda$, then $F(x_1, w^\star(x_1), D\lambda(x_1)) \leq 0$.

Let $\overline{\text{co}}(S)$ denote the closed convex hull of any set $S \subseteq \mathbb{R}^M$. We will set $\Phi(x, A) := \{\Phi(x,a) : a \in A\}$ for all $\Phi : \mathbb{R}^N \times A \to \mathbb{R}^M$, and $\text{cl}(S)$ (resp., $\text{comp}(S)$) will denote the topological closure of S (resp., $\mathbb{R}^N \setminus S$) for each $S \subseteq \mathbb{R}^N$. For any function $\Psi : \mathbb{R}^N \to [-\infty, +\infty]$, we define the **(effective) domain** of Ψ to be $\text{dom}(\Psi) = \{x \in \mathbb{R}^N : \Psi(x) \in \mathbb{R}\}$, and we define $\text{Trace}(\Psi) := \{\Psi(t) : t \in \text{dom}(\Psi)\}$. Also, $\partial(B)$ denotes the boundary of any set $B \subseteq \mathbb{R}^N$, and $\text{Null}(F) := \{x : F(x) = 0\}$ for any real-valued function F. A set $\mathcal{G} \subseteq \mathbb{R}^N$ will be called **(strongly) invariant** with respect to f (or f-invariant) provided the following holds: If $x \in \mathcal{G}$, then $\phi(t) \in \mathcal{G}$ for all $t > 0$ and $\phi \in \text{Traj}_x(f)$. An open f-invariant set $\mathcal{O}$ containing the origin is called **asymptotically null (for f)** provided the following holds: If $x \in \mathcal{O}$ and $\phi \in \text{Traj}_x(f)$, then $\phi(t) \to 0$ as $t \to +\infty$. One can show that $\mathcal{D}_o$ is asymptotically null for f when f is ULES. Moreover, we have the following, whose proof is in [6]:

Lemma 2.3 Let f be a ULAS dynamics satisfying (A_1)-(A_3). Then:

a) $\mathrm{cl}(B_r) \subseteq \mathcal{D}_o$, and $\mathcal{D} \subseteq \mathrm{cl}(\mathcal{D}_o)$.

b) $\mathcal{D}_o$ is open and asymptotically null for f.

c) If $f(x, A)$ is convex for all $x \in \mathbb{R}^N$, then $\mathcal{D} = \mathcal{D}_o$.

For each $x \in \mathbb{R}^N$, $\alpha \in \mathcal{A}^r$, and h and ℓ satisfying (A_4)-(A_5), we set

$$\tilde{h}(x, t, \alpha) := \int_0^t h^r(\phi(s, x, \alpha), \alpha(s))\,\mathrm{d}s$$

and

$$J[\ell, h](x, t, \alpha) := \int_0^t e^{-\tilde{h}(x,s,\alpha)} \ell^r(\phi(s, x, \alpha), \alpha(s))\,\mathrm{d}s$$

For cases where $h \equiv 0$, we use $J[\ell]$ to signify $J[\ell, 0]$. We also set

$$\mathrm{dist}(p, S) = \inf\{||p - s|| : s \in S\}$$

and $B_R(S) := \{x \in \mathbb{R}^N : \mathrm{dist}(x, S) \leq R\}$ for each $R > 0$, $p \in \mathbb{R}^N$, and $S \subseteq \mathbb{R}^N$. When $S = \{\bar{x}\}$, we write $B_R(\bar{x})$ instead of $B_R(\{\bar{x}\})$. The following lemma follows from the proof of Theorem III.2.32 in [3]:

Lemma 2.4 Let (A_1)-(A_5) hold, and w be a solution of (8) on a bounded open set $B \subseteq \mathbb{R}^N$. Define $\tau_q : \mathcal{A} \to [0, \infty]$ and $T_\delta : \mathbb{R}^N \to [0, \infty]$ by

$$\begin{aligned} \tau_q(\beta) &= \inf\{\, t \geq 0 : \ \phi(t, q, \beta) \ \in \partial B\} \\ T_\delta(p) &= \inf\{t \geq 0 : \phi(t, p, \alpha) \in B_\delta(\partial B), \alpha \in \mathcal{A}\} \end{aligned} \tag{9}$$

for each $q \in B$ and $\delta > 0$. Then:

(a) For all $q \in B$, $\beta \in \mathcal{A}$, and $r \in [0, \tau_q(\beta))$,

$$w^\star(q) \ \leq \ \int_0^r e^{-\tilde{h}(q,s,\beta)} \ell(\phi(s, q, \beta), \beta(s))\,\mathrm{d}s \ + \ e^{-\tilde{h}(q,r,\beta)} w^\star(\phi(r, q, \beta)).$$

(b) For all $q \in B$, $\delta \in (0, \mathrm{dist}(q, \partial B)/2)$, and $t \in [0, T_\delta(q))$,

$$\begin{aligned} w_\star(q) \ \ \geq \ \ \inf_{\alpha \in \mathcal{A}} & \left[\int_0^t e^{-\tilde{h}(q,s,\alpha)} \ell(\phi(s, q, \alpha), \alpha(s))\,\mathrm{d}s \right. \\ + & \left. \ e^{-\tilde{h}(q,t,\alpha)} w_\star(\phi(t, q, \alpha)) \right]. \end{aligned}$$

A function $V : \mathcal{O} \to \mathbb{R}$ on an open set $\mathcal{O} \subseteq \mathbb{R}^N$ is called a **robust Lyapunov function** for f provided (i) V is positive definite (meaning, $V(x) \geq 0$ for all x, and $V(x) = 0$ iff $x = 0$) and (ii) $V(x) > V(\phi(t, x, \alpha))$ for all $x \in \mathcal{O} \setminus \{0\}$, $t > 0$, and $\alpha \in \mathcal{A}$. An open set $\mathcal{O} \subseteq \mathbb{R}^N$ is called g **quasi-stable** (resp., **relaxed** g **quasi-stable**) provided the following condition holds for each $x \in \mathcal{O}$: If $\alpha \in \mathcal{A}$ and $\int_0^\infty g(\phi(t, x, \alpha), \alpha(t))\,\mathrm{d}t < \infty$ (resp., $\alpha \in \mathcal{A}^r$ and $\int_0^\infty g^r(\phi(t, x, \alpha), \alpha(t))\,\mathrm{d}t < \infty$), then $\lim_{t \to +\infty} \phi(t, x, \alpha) = 0$.

Remark 2.5 If we drop the assumption that f is bounded but keep the other hypotheses in (A_1)-(A_3) the same, then (5) has a unique solution $\phi(\cdot, x_o, a)$ defined on a maximal interval $[0, b)$, with $b > 0$ depending on x_o and $a \in \mathcal{A}$. Recall that if A is a compact metric space, then a uniformly locally Lipschitz function $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ (cf. (A_2)) is called **forward complete** provided that the solution $\phi(\cdot, x_o, a)$ for (5) is defined on $[0, \infty)$ for all $x_o \in \mathbb{R}^N$ and $a \in \mathcal{A}$ (cf. [12]). Assumption (A_2) implies that f is forward complete. As mentioned in [8], it is not very restrictive to assume that f is bounded, since we can replace f by $f/(1+||f||)$ without changing the trajectories. However, normalizing in this way changes the design of the control system. If we change (A_1)-(A_3) by replacing the boundedness of f with the assumption that f is forward complete, then $\mathcal{R}^T(S) := \{\phi(t, x, \alpha) : 0 \leq t \leq T, x \in S, \alpha \in \mathcal{A}\}$ is bounded for each $T \in [0, \infty)$ and bounded set $S \subseteq \mathbb{R}^N$ (cf. [12]). Using this fact, one can show that all results in this note remain true if the boundedness of f is replaced by the forward completeness of f (and the other hypotheses are kept the same). The system f will be forward complete if, instead of assuming f is bounded, we assume $f(x, a)$ is globally Lipschitz in x uniformly in a (cf. [3]).

Remark 2.6 It will not be necessary to assume that the sets $f(x, A)$ are convex. Instead, we will instead apply Lemma 2.3(c) to the *relaxed* dynamics $f^r : \mathbb{R}^N \times A^r \to \mathbb{R}^N$, using the fact that $f^r(x, A^r) \equiv \overline{\text{co}}(f(x, A))$ is convex for all $x \in \mathbb{R}^N$. In §3, we study solutions of (8) on general asymptotically null sets, including non-ULAS f (cf. [9], p. 191, for an example due to Vinograd of asymptotically null sets for non-ULAS dynamics, with no controls, in which each trajectory asymptotically approaches the origin).

Remark 2.7 By the Filippov Selection Theorem, relaxed g quasi-stability and g quasi-stability are equivalent if $(f \times g)(x, A)$ is convex for each $x \in \mathbb{R}^N$ (cf. [3], §VI.1). For sufficient conditions for g quasi-stability for cascade systems, see [16, 17]. Notice that both forms of g quasi-stability hold for all choices of g if $\mathcal{O}$ is asymptotically null for f^r. Also, both are satisfied for $\mathcal{O} = \mathbb{R}^N$ and ULES f if (2) holds for the constant r in the ULES definition. Since $\mathcal{A} \subseteq \mathcal{A}^r$, relaxed g quasi-stability implies g quasi-stability for any open set. In some applications, we also assume g is uniformly locally Lipschitz (cf. (A_6) in §4 below), in which case g quasi-stability and relaxed g quasi-stability are equivalent conditions for any open $\mathcal{O} \subseteq \mathbb{R}^N$ (cf. Appendix A below).

3 Results on Infinite Horizon Problems

To develop Lyapunov theory for ULAS dynamics, we first study the infinite horizon value function

$$V_\infty(x) := \inf_{\alpha \in \mathcal{A}} J[\ell, h](x, +\infty, \alpha) \quad \in \quad [-\infty, +\infty] \tag{10}$$

with the conventions that (i) the infimum is only over those α for which $J[\ell, h](x, +\infty, \alpha)$ converges in $\mathbb{R} \cup \{\pm\infty\}$ and (ii) $\inf \emptyset = +\infty$. Our results on (10) will be of independent interest, because we will allow general continuous ℓ, unlike the usual results, which assume that $\ell \geq 0$ everywhere, or that h is uniformly bounded below by a positive constant (cf. the remarks following the proof of Theorem 1 for a comparison of our results with the known results). Since minimization of a function F is equivalent to maximizing $-F$, this allows problems where minimization takes place in the part of the state space where $\ell \geq 0$, while maximization takes place elsewhere. We will refer to the side condition

(SC_w) $\quad w(0) = 0$, and w is continuous at the origin

In particular, (SC_w) implies that $w_\star(0) = w^\star(0) = 0$, and that for each $\varepsilon > 0$, there exists a $\delta > 0$ for which $w^\star(p) < \varepsilon$ and $w_\star(p) > -\varepsilon$ for all $p \in B_\delta$. We will use this information in the proofs of Theorem 1, Theorem 2, and Proposition A.2 below, to account for the case where w is discontinuous.

3.1 Problems with Negative Lagrangians

We begin with the case of *nonpositive* ℓ, in which case (10) involves the maximization of $J[-\ell, h]$. For the extension to *general* Lagrangians, and to sets which are not asymptotically null, see Remark 3.1 and §3.2. Notice that V_∞ might not be continuous or even locally bounded, and so not a solution of (8) (cf. Remark 3.1, with $A = \{+1\}$). Nevertheless, we have the following local comparison result, whose statement and proof actually cover the cases where ℓ is everywhere nonpositive or everywhere nonnegative:

Theorem 1 *Assume the following:*

(1) (A_1),(A_2), (A_4), (A_5), $\mathrm{Null}(||f(0,\cdot)||) \neq \emptyset$.

(2) $\mathcal{G} \subseteq \mathbb{R}^N$ *is asymptotically null for* f.

(3) Either $\ell(x,a) \leq 0$ *for all* $(x,a) \in \mathbb{R}^N \times A$, *or else* $\ell(x,a) \geq 0$ *for all* $(x,a) \in \mathbb{R}^N \times A$.

(4) $w : \mathcal{G} \to \mathbb{R}$ *is a solution of (8) on* $\mathcal{G} \setminus \{0\}$ *that satisfies* (SC_w).

Then $w \equiv V_\infty$ *on* $\mathcal{G}$.

Proof. We assume $h \equiv 0$. The proof of the general case is similar to the one we will now give. Let $\bar{x} \in \mathcal{G} \setminus \{0\}$ and $\varepsilon > 0$ be given. It suffices to check that $w^\star(\bar{x}) \leq V_\infty(\bar{x})$ and $w_\star(\bar{x}) \geq V_\infty(\bar{x})$, since $w_\star \leq w^\star$ on $\mathcal{G}$. To do this, first note that $J[\ell](\bar{x}, +\infty, \alpha)$ converges in $\mathbb{R} \cup \{\pm\infty\}$ for all $\alpha \in \mathcal{A}$, by hypothesis (3). If $\alpha \in \mathcal{A}$, then Lemma 2.4, the asymptotic nullness of $\mathcal{G}$, and (SC_w) give

$$\begin{aligned} w^\star(\bar{x}) \quad &\leq \quad \limsup_{t \to +\infty} \left[\int_0^t \ell(\phi(s,\bar{x},\alpha), \alpha(s))\, \mathrm{d}s + w^\star(\phi(t,\bar{x},\alpha)) \right] \\ &\leq \quad J[\ell](\bar{x}, +\infty, \alpha) + \limsup_{t \to +\infty} w^\star(\phi(t,\bar{x},\alpha)) \quad \leq \quad J[\ell](\bar{x}, +\infty, \alpha) \end{aligned}$$

so $w^\star(\bar{x}) \le V_\infty(\bar{x})$ follows by infimization. Let $\varepsilon > 0$ be given. It remains to construct $\hat{\alpha} \in \mathcal{A}$ such that

$$w_\star(\bar{x}) \;\ge\; \int_0^M \ell(\phi(t,\bar{x},\hat{\alpha}),\hat{\alpha}(t))\,\mathrm{d}t \;+\; w_\star(\phi(M,\bar{x},\hat{\alpha})) - \varepsilon \quad \forall M \in \mathbb{N} \tag{11}$$

Since $w_\star$ is lower semicontinuous and $\mathcal{G}$ is asymptotically null, a passage to the liminf as $M \to +\infty$ and an infimization in (11) would then give $w_\star(\bar{x}) \ge V_\infty(\bar{x}) - \varepsilon$. Since ε was arbitrary, we would get $w_\star(\bar{x}) \ge V_\infty(\bar{x})$, as needed. We will now construct an $\hat{\alpha}$ satisfying (11) by adapting ideas from [16, 17]. We will assume that w is continuous on $\mathcal{G}$, the general case being exactly the same but with $w_\star$ replacing w. Define $E_1, E_2, \ldots$ by

$$E_j(t) \;\equiv\; \varepsilon\left[e^{-(j-1)} - e^{-(t+j-1)}\right] \;\text{ for each }\; t > 0 \tag{12}$$

By the choice of the E_j's,

$$E_1(1) + E_2(1) + \ldots + E_m(1) = \varepsilon(1 - e^{-m})$$

for all $m \in \mathbb{N}$. Set

$$\mathcal{Z}_1 \;:=\; \left\{ \begin{array}{l} (t,\alpha) \in [0,1] \times \mathcal{A}:\; w(\bar{x}) \ge \int_0^t \ell(\phi(s,\bar{x},\alpha),\alpha(s))\,\mathrm{d}s \\ +w(\phi(t,\bar{x},\alpha)) - E_1(t) \end{array} \right\}$$

The proof of the theorem in [17], which we review in Appendix A, shows that $\mathcal{Z}_1$ contains an element of the form $(1, \bar{\alpha}^{(1)})$. Set

$$c_1 \equiv \bar{\alpha}^{(1)}, \;\; \bar{\gamma}(0) = \bar{x}, \;\; \bar{\gamma}(1) = \phi(1,\bar{x},c_1).$$

Next, we will inductively define the sets

$$\mathcal{Z}_{i+1} := \left\{ \begin{array}{l} (t,\alpha) \in [0,1] \times \mathcal{A}:\; w(\bar{\gamma}(i)) - w(\phi(t,\bar{\gamma}(i),\alpha)) \ge \\ \int_0^t \ell(\phi(s,\bar{\gamma}(i),\alpha),\alpha(s))\,\mathrm{d}s - E_{i+1}(t) \end{array} \right\} \tag{13}$$

Reapplying the argument from [17], we inductively obtain

$$(1, \bar{\alpha}^{(i)}) \in \mathcal{Z}_i \;\text{ and }\; \bar{\gamma}(i+1) := \phi(1, \bar{\gamma}(i), \bar{\alpha}^{i+1}) \in \mathcal{G}$$

for all $i \in \mathbb{N}$. We inductively let c_{i+1} be the concatenation of $c_i\lceil[0,i]$ followed by $\bar{\alpha}^{(i+1)}$. Now sum both sides of the inequality in (13), with the choices $\alpha = \bar{\alpha}^{i+1}$ and $t = 1$, over $i = 0, 1, 2, \ldots, M-1$. Since

$$\sum_{i=0}^{M-1} J[\ell](\bar{\gamma}(i), 1, \bar{\alpha}^{i+1}) = J[\ell](\bar{x}, M, c_M)$$

for all $M \in \mathbb{N}$, the choices of the E_j's give (11), with $\hat{\alpha}(t) := c_j(t)$ for $0 \le t \le j$. ∎

Remark 3.1 Notice that no growth conditions were required for ℓ, nor were there any requirements on the *rate* the trajectories approach the origin. In particular, we allow non-Lipschitz ℓ and unstable f (cf. Remark 2.6). Condition (3) on ℓ in Theorem 1 was used to force the integral in (11) to converge in $\mathbb{R}\cup\{\pm\infty\}$. However, for general ℓ, $J[\ell](\bar{x}, +\infty, \hat{\alpha})$ may not converge in $\mathbb{R}$. This occurs if for example

$$\begin{array}{l} N = 1, \quad 1 \in A, \quad f(x,1) \equiv -x^3, \\ h \equiv 0, \quad \ell(x,a) \equiv |x|, \quad \hat{\alpha} \equiv 1 \end{array}$$

in which case

$$J[\ell](1, M, \hat{\alpha}) = \int_0^M (1+2t)^{-1/2}\mathrm{d}t,$$

which does not converge in $\mathbb{R}$ as $M \to +\infty$. A different way to guarantee convergence of the costs is to replace Condition (3) in Theorem 1 with one of the following:

(A) ℓ is bounded, $\exists h_o > 0$ s.t. $h(x,a) \geq h_o \ \forall (x,a) \in \mathbb{R}^N \times A$.

(B) $|\ell(x,a)| \leq h(x,a) \ \forall (x,a) \in \mathbb{R}^N \times A$.

In case (B), we get

$$J[|\ell|, h](x, +\infty, \alpha) \leq J[h,h](x,+\infty,\alpha) = 1 - \exp\{-J[h](x,+\infty,\alpha)\} \leq 1$$

for all $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}$. For case (A), uniqueness characterizations for (8) are known (cf. [3], Chapter III). On the other hand, since we allow general h and ℓ, case (B) is not covered by the usual uniqueness results. In both cases, $J[\ell, h](x, +\infty, \alpha) \in \mathbb{R}$ for all $x \in \mathcal{G}$ and $\alpha \in \mathcal{A}$ and the argument we used to prove Theorem 1 applies. Further extensions of Theorem 1 are given in §3.2.

Remark 3.2 If we add the assumption in Theorem 1 that f is ULES, then the part of the proof of Theorem 1 showing that $w_\star(\bar{x}) \geq V_\infty(\bar{x})$ can be simplified to the following. First assume $\ell(x,a) \leq 0$ for all $x \in \mathbb{R}^N$ and $a \in A$. Let r be as in the ULES definition, let $\varepsilon > 0$ be given, and choose $\delta \in (0, r)$ such that $w_\star(p) > -\frac{\varepsilon}{2}$ for all $p \in B_\delta$ (using the lower semicontinuity of $w_\star$ and (SC_w)). For convenience, we will assume that the constant C in the ULES definition is 1, but a similar argument to the one we will now give applies if $C > 1$, by considering small overshoots. By $(\star\star)$ (with $\beta_f(s,t) = se^{-\sigma t}$ for suitable $\sigma > 0$), B_δ is then f-invariant. Set

$$\tau(p, \mu, \alpha) = \inf\{t \geq 0 : ||\phi(t, p, \alpha)|| \leq \mu\}$$

for all $p \in \mathcal{G}$, $\mu > 0$, and $\alpha \in \mathcal{A}$, and let $\mathcal{N} \subseteq \mathcal{G}$ be an open neighborhood of $\bar{x}$. If

$$\bar{T} := \sup\{\tau(\bar{x}, \delta, \alpha) : \alpha \in \mathcal{A}\} = +\infty,$$

then by Lemma III.2 from [20], there exist $q \in \mathcal{N}$ and $\beta \in \mathcal{A}$ such that

$$\tau(q, \delta/2, \beta) = +\infty.$$

This contradicts the asymptotically nullness of $\mathcal{G}$. It follows that $\bar{T} < \infty$. By the convergence of all costs $J[\ell](x, +\infty, \alpha)$, it suffices to show that

$$\mathcal{Z}_{\bar{T}} \;:=\; \left\{ \begin{array}{l} (t, \alpha) \in [0, \bar{T}] \times \mathcal{A} : w_\star(\bar{x}) \geq \int_0^t \ell(\phi(s, \bar{x}, \alpha), \alpha(s))\,\mathrm{d}s \\ +w_\star(\phi(t, \bar{x}, \alpha)) - \frac{\varepsilon}{2}(1 - e^{-t}) \end{array} \right\}$$

contains a pair $(\bar{T}, \bar{\alpha})$, since then

$$w_\star(\bar{x}) \geq V_\infty(\bar{x}) - \varepsilon.$$

The proof that $\mathcal{Z}_{\bar{T}}$ contains such a pair is similar to the argument of the theorem in [17] (cf. Appendix A below). A similar simplification can be made if, instead of assuming that ℓ is nonpositive, we assume $0 \leq \ell(x, a) \leq ||x||$ for all (x, a) (which implies that V_∞ is continuous at the origin).

Remark 3.3 Whereas the results of [17] assume the controllability condition $STC\{0\}$ to the origin (cf. [3]), the previous theorem does not require any controllability. For example, it applies to the ULES dynamics

$$f(x, a) = (-x_1, -x_2) + a(x_1^2, x_2^2),\ A = [-1, +1],$$

on $\mathcal{G} = (-1, +1)^2$, where $STC\{0\}$ is not satisfied. In the usual PDE characterizations for (8) (cf. [3, 13, 14, 17]), one also assumes that the comparison functions w are either bounded-from-below or that $w(x) \to +\infty$ as $x \to \partial\mathcal{G}$. However, these assumptions are not needed in Theorem 1. Therefore, Theorem 1 can be regarded as an extension of the earlier results which takes a larger class of possible dynamics and solutions into account.

Remark 3.4 Note that it was not necessary to assume any uniform bounds on the Lagrangians. A uniform positive lower bound on ℓ or h is needed in the uniqueness characterizations of [3, 6, 21]. Therefore, Theorem 1 also extends the earlier results by allowing more general Lagrangians and interest rates. Recall that V_∞ may not be locally bounded (cf. Remark 3.1, with $A = \{+1\}$). However, the arguments of [6] establish that if (A_1)-(A_4) hold with f ULES, $h \equiv 0$, and $\ell = -g$ for g uniformly locally Lipschitz (cf. (A_6) in §4 below), then $w = V_\infty$ is a solution of (8) on $\mathcal{D}_o$ that satisfies (SC_w). Since $\mathcal{D}_o$ is asymptotically null for f, Theorem 1 then gives a PDE characterization for V_∞ on $\mathcal{D}_o$.

3.2 Problems with General Lagrangians

By combining the arguments of Theorem 1 with [17], the following variant of Theorem 1 is easily shown:

Proposition 3.5 Assume the following:

(1) (A_1), (A_2), (A_4), $h \equiv 0$, $\mathrm{Null}(||f(0,\cdot)||) \neq \emptyset$.

(2) ℓ is nonnegative, $\mathcal{G} \subseteq \mathbb{R}^N$ is ℓ quasi-stable and f-invariant.

(3) $w : \mathcal{G} \to \mathbb{R}$ is a bounded-from-below solution of (8) on $\mathcal{G} \setminus \{0\}$ that satisfies (SC_w).

Then $w \equiv V_\infty$ on $\mathcal{G}$.

Proposition 3.5 applies to the Fuller Example data from [13, 14] on $\mathcal{G} = \mathbb{R}^2$ (in which $N = 2$, $A = [-1,+1]$, $h \equiv 0$, $f(x,a) = (x_2, a)$, $\ell(x,a) \equiv |x_1|^\gamma$, $\gamma > 1$), which is not tractable by means of the usual results on first-order viscosity solutions. For a discussion of this application, see [13, 16, 17]. (For a different uniqueness characterization that applies to the Fuller Example, as well as to other examples in which the quasi-stability condition in (2) of Proposition 3.5 is not satisfied, see [15].)

The nonpositivity of ℓ in Theorem 1 was used to guarantee that the total cost $J[\ell,h](\bar{x},+\infty,\hat{\alpha})$ for the constructed input $\hat{\alpha}$ converged in $\mathbb{R} \cup \{\pm\infty\}$. This convergence is implied by (1)-(3) in Proposition 3.5. A totally different approach to guaranteeing this convergence, which allows general continuous ℓ, is as follows. Set

$$V_\infty^r(x) := \inf_{\alpha \in \mathcal{A}^r} J[\ell,h](x,+\infty,\alpha) \quad \in \quad [-\infty,+\infty], \tag{14}$$

with the same conventions (i)-(ii) used to define V_∞ in (10).

Proposition 3.6 Assume (A_1)-(A_5) hold and $\mathcal{G}$ is asymptotically null for f^r. Let $w : \mathcal{G} \to \mathbb{R}$ be a continuous solution of (8) on $\mathcal{G} \setminus \{0\}$ satisfying $w(0) = 0$. Then $w \equiv V_\infty^r$ on $\mathcal{G}$.

Proof. We show how to modify the proof of Theorem 1. We again assume $h \equiv 0$, the general case being handled in a similar way. Let $\bar{x} \in \mathcal{G} \setminus \{0\}$. For each $\varepsilon > 0$, there exists $(1, \alpha_\varepsilon) \in \mathcal{Z}_1$ such that

$$\begin{aligned} w(\bar{x}) \quad \geq \quad & \int_0^1 \ell(\phi(s,\bar{x},\alpha_\varepsilon), \alpha_\varepsilon(s))\,\mathrm{d}s \\ + \quad & w(\phi(1,\bar{x},\alpha_\varepsilon)) - \varepsilon \geq w(\bar{x}) - \varepsilon \end{aligned} \tag{15}$$

with the last inequality following from the first part of Lemma 2.4. Using Lemma 2.1 with $c = 1$, we can find a subsequence of the α_ε's (which we do not relabel) and $\beta \in \mathcal{A}^r$ such that (i) $\alpha_\varepsilon \to \beta$ weak-$\star$ on $[0,1]$ and (ii) $\max\{||\phi(t,\bar{x},\alpha_\varepsilon) - \phi(t,\bar{x},\beta)|| : 0 \leq t \leq 1\} \to 0$ as $\varepsilon \downarrow 0$. Since $\mathcal{G}$ is f^r-invariant, it follows that $\phi(t,\bar{x},\beta) \in \mathcal{G}$ for all $t \in [0,1]$. Letting $\varepsilon \downarrow 0$ in (15) and using the continuity of w now gives

$$w(\bar{x}) = \int_0^1 \ell^r(\phi(s,\bar{x},\beta), \beta(s))\,\mathrm{d}s + w(\phi(1,\bar{x},\beta))$$

The proof of the preceding equality is based on the continuity of the maps $a \mapsto \ell(x,a)$ for all x, and is similar to the argument in the appendix of [13]. This procedure is iterated and gives $\hat{\alpha} \in \mathcal{A}^r$ such that

$$w(\bar{x}) = \int_0^M \ell^r(\phi(s,\bar{x},\hat{\alpha}),\hat{\alpha}(s))\,\mathrm{d}s + w(\phi(M,\bar{x},\hat{\alpha})) \quad \forall M \in \mathbb{N} \tag{16}$$

Since $\mathcal{G}$ is asymptotically null for f^r and w is continuous, $w(\phi(M,\bar{x},\hat{\alpha})) \to 0$ as $M \to +\infty$. This implies that the integral in (16) converges in $\mathbb{R}$ as $M \to +\infty$, so (16) gives $w(\bar{x}) \geq V^r_\infty(\bar{x})$. The proof that $w(\bar{x}) \leq V^r_\infty(\bar{x})$ is similar to the proof of the corresponding inequality in the proof of Theorem 1, with $\mathcal{A}$ replaced by $\mathcal{A}^r$, since (cf. [3])

$$\sup\{-f^r(x,a)\cdot p - \ell^r(x,a) : a \in A^r\} \;=\; \sup\{-f(x,a)\cdot p - \ell(x,a) : a \in A\}$$

for all $x \in \mathbb{R}^N$. ∎

Remark 3.7 In some applications, we will take f ULES, $h \equiv 0$, and $\ell = -g$ for uniformly locally Lipschitz g (cf. (A_6) below). In this case, V_∞ is a continuous solution of (8) on $\mathcal{D}^r_o \setminus \{0\}$ (with $h \equiv 0$), where $\mathcal{D}^r_o$ is the robust domain of attraction for $f^r : \mathbb{R}^N \times A^r \to \mathbb{R}^N$.[5] Proposition 3.6 then implies that $V_\infty \equiv V^r_\infty$ on $\mathcal{D}^r_o$. We will show below that we in fact have $V_\infty \equiv V^r_\infty$ on all of $\mathcal{D}_o$, since $\mathcal{D}_o = \mathcal{D}^r_o$.

4 Properties of Domains of Attraction and Lyapunov Functions

Our cost function g gives rise to value functions, which we denote by V_L, for infinite horizon cost maximization (cf. (18)). In this section, we add hypotheses on g which imply that V_L is a robust Lyapunov function for f, and that V_L is a unique solution of the PDE (4). We also give PDE characterizations for $\mathcal{D}_o$. Regularity for V_L is covered in §4.2. In particular, we show that by removing the positive lower bound assumption (2) on g, we obtain Lyapunov functions with special properties not found in the Lyapunov functions of [4, 5, 6]. In §4.3, these results are used to give uniqueness characterizations for solutions of the generalized Zubov equation (1), giving a new, more general class of robust Lyapunov functions which can be characterized as unique solutions of generalized Zubov equations.

4.1 Domains of Attraction

To study $\mathcal{D}_o$, we first consider the *auxiliary* PDE

$$\inf_{a\in A}\{-f(x,a)\cdot Dv(x) - g(x,a)\} = 0 \tag{17}$$

[5] Recall that the *relaxed domains of attraction* are defined in the following way: $\mathcal{D}^r := \left\{x \in \mathbb{R}^N : \lim_{t\to+\infty} \phi(t,x,\alpha) = 0\ \forall \alpha \in \mathcal{A}^r\right\}$, and $\mathcal{D}^r_o := \left\{x \in \mathbb{R}^N : \sup_{\alpha\in\mathcal{A}^r} t(x,\alpha) < +\infty\right\}$.

under hypotheses (A_1)-(A_4). It is immediate from the definition of viscosity solutions that a function w is a viscosity solution of (17) on an open set $\mathcal{O}$ exactly when $-w$ is a viscosity solution of (8) on $\mathcal{O}$, with the choices $\ell = -g$ and $h \equiv 0$. Equation (8) is the usual Hamilton-Jacobi equation, which was studied for general nonnegative ℓ in [13, 14]. However, it will be convenient for our applications to assume that g is nonnegative, so the earlier uniqueness results on (8) for nonnegative ℓ and $h \equiv 0$ would not apply. Instead, we study (17) using the theory of the previous section. We set

$$V_L(x) = \sup_{a \in \mathcal{A}} J[g](x, +\infty, a) \in [0, +\infty], \quad x \in \mathbb{R}^N \tag{18}$$

We sometimes write $V_L[g]$ instead of V_L, to emphasize the cost function g. If (A_1)-(A_4) hold with f ULES, and if

$$\exists\ \tilde{C},\ \lambda > 0 \text{ s.t. } g(x, a) \leq \tilde{C}||x||^\lambda \text{ for all } x \in B_r \text{ and } a \in A \tag{19}$$

then the argument of [6] shows that $V_L[g]$ is locally bounded on $\mathcal{D}_o$ (i.e., $\sup\{V_L(x) : x \in K\} < \infty$ for each compact set $K \subseteq \mathcal{D}_o$). Moreover, if g satisfies the *stronger condition*

(A_6) For each $R > 0$, there exists a positive constant $L_{g,R}$ such that $|g(x, a) - g(y, a)| \leq L_{g,R}||x - y||$ for all $x, y \in B_R$ and $a \in A$.

then $V_L[g]$ is continuous on $\mathcal{D}_o$ (cf. [6]). The following theorem extends the uniqueness result in [6], Theorem 3.9, to the case of general nonnegative cost functions:

Theorem 2 *Assume (A_1)-(A_4) and (19) hold and f is ULES. Let $\mathcal{O} \subseteq \mathbb{R}^N$ be a g quasi-stable set containing the origin, and let $w : \mathcal{O} \to \mathbb{R}$ be a bounded-from-below solution of (17) on $\mathcal{O}$ satisfying condition (SC_w) and $w(x) \to +\infty$ as $x \to x_o$ for all $x_o \in \partial\mathcal{O}$. Then $\mathcal{O} = \mathcal{D}_o$, and $w \equiv V_L$ on $\mathcal{D}_o$.*

Proof. We can assume that w is continuous on $\mathcal{O}$. (Indeed, in what follows, we do not use the full strength of the continuity of w. Instead, we only use the lower semicontinuity of $w_\star$ and the upper semicontinuity of $w^\star$.) As already noted, $-w$ is a solution of (8) with $\ell = -g$ and $h \equiv 0$ on $\mathcal{O}$. It follows from Lemma 2.4 and the boundary behavior of w that $\mathcal{O}$ is f-invariant. Indeed, if $x \in \mathcal{O}$ and $\alpha \in \mathcal{A}$, and if t is the first time $\phi(t, x, \alpha) \in \partial\mathcal{O}$, then we can apply the first part of Lemma 2.4 to (8), with $q = x$, $\ell = -g$, $h \equiv 0$, and $\beta = \alpha$, and with bounded open sets $B \subseteq \mathcal{O}$ containing $\text{Trace}(\phi(\cdot, x, \alpha)\lceil[0, t - 1/m])$ for $m \in \mathbb{N}$, to get

$$-w(x) \leq -\int_0^{t-1/m} g(\phi(s, x, \alpha), \alpha(s))\, \mathrm{d}s - w(\phi(t - 1/m, x, \alpha)) \to -\infty$$

as $m \to +\infty$, which is a contradiction. Also, the first part of Lemma 2.4 and the invariance of $\mathcal{O}$ then give

$$w(x) \geq \int_0^t g(\phi(s, x, \alpha), \alpha(s))\, \mathrm{d}s + w(\phi(t, x, \alpha)) \tag{20}$$

for all $\alpha \in \mathcal{A}$, $x \in \mathcal{O}$, and $t > 0$. Since w is bounded-from-below and g is nonnegative, (20) gives $J[g](x, +\infty, \alpha) < \infty$ for all $\alpha \in \mathcal{A}$ and $x \in \mathcal{O}$. The g quasi-stability of $\mathcal{O}$ and Lemma 2.3 therefore give $\mathcal{O} \subseteq \mathcal{D} \subseteq \mathrm{cl}(\mathcal{D}_o)$. Since $\mathcal{O}$ is open, $\mathcal{O} \subseteq \mathcal{D}_o$. Since $\mathcal{D}_o$ is asymptotically null for f and $\mathcal{O}$ is an open invariant set containing the origin, $\mathcal{O}$ is also asymptotically null for f. Therefore, Theorem 1 (with the choices $\ell = -g$ and $h \equiv 0$) gives $w \equiv V_L$ on $\mathcal{O}$. Therefore, $V_L(x) \to +\infty$ as $\mathcal{O} \ni x \to x_o$ for all $x_o \in \partial\mathcal{O}$. By (19), V_L is locally bounded on $\mathcal{D}_o$, so we conclude that $\mathcal{D}_o \cap \partial\mathcal{O} = \emptyset$. Recall that $\mathcal{O}$ contains a neighborhood of the origin. If $p_o \in \mathcal{D}_o \setminus \mathcal{O}$, then the asymptotic nullness of $\mathcal{D}_o$ gives a point $p_1 \in \mathcal{D}_o \cap \partial\mathcal{O}$, namely, the first point on a trajectory for f starting at p_o and approaching the origin that lies in $\partial\mathcal{O}$. This contradiction gives $\mathcal{D}_o \subseteq \mathcal{O}$ and completes the proof. ∎

Notice that *any* open set $\mathcal{O} \subseteq \mathbb{R}^N$ is g quasi-stable if (i) f is ULES and (ii) g satisfies (2) for the constant r in the ULES definition, which is the lower bound requirement in [4, 5, 6]. Therefore, the uniqueness results for (17) given in [6] all follow from Theorem 2. Moreover, the comparison results for (17) given in [6] also require the comparison functions w to be proper (meaning $w(x) \to +\infty$ as $||x|| \to \infty$) and (A_6). Therefore, Theorem 2 *extends* the uniqueness results in [6] by taking a more general class of comparison functions into account. Moreover, Theorem 2 applies to more general g, including non-Lipschitz g that do not satisfy (2) (cf. §5). If (A_1)-(A_4) and (A_6) hold, with f ULES, then V_L is a continuous solution of (17) on $\mathcal{D}_o$ by arguments in [6], so Theorem 2 gives PDE characterizations for V_L and $\mathcal{D}_o$ if $V_L(x) \to +\infty$ as $x \to \partial(\mathcal{D}_o)$. Sufficient conditions for this boundary behavior of V_L will be given in the next subsection.

4.2 Further Properties of V_L

In this section, we study the behavior of V_L near $\partial(\mathcal{D}_o)$ and the regularity of V_L. If we assume (A_1)-(A_4) and (A_6) with f ULES, and if we also assume

(A_7) $\int_0^t g(\phi(s, x, a), a(s))\,\mathrm{d}s > 0$ for all $x \in \mathcal{D}_o \setminus \{0\}$, $t > 0$, and $a \in \mathcal{A}$

then V_L is a robust Lyapunov function for f on $\mathcal{D}_o$. Indeed, the dynamic programming methods from [3] give

$$V_L[g](x) = \sup_{\alpha \in \mathcal{A}} \left[\int_0^t g(\phi(s, x, \alpha), \alpha(s))\,\mathrm{d}s + V_L[g](\phi(t, x, \alpha)) \right] \quad \forall t > 0 \quad (21)$$

for all $x \in \mathcal{D}_o$, which gives $V_L(x) > V_L(\phi(t, x, \alpha))$ for all $x \in \mathcal{D}_o \setminus \{0\}$, $t > 0$, and $\alpha \in \mathcal{A}$, as needed. Since Lyapunov functions form the basis for much of current work in stability, this motivates our study of further properties of V_L. Notice that (A_7) is weaker than (2) (cf. Remark 4.9 and §5 below for examples).

4.2.1 Boundary Behavior of V_L

We will assume (A_1)-(A_4) and (A_6)-(A_7), with f ULES, throughout the remainder of this subsection. We first look for conditions under which

$$V_L(x) \ \rightarrow \ +\infty \ \text{ as } \ x \rightarrow x_o \ \text{ for all } \ x_o \in \partial(\mathcal{D}_o) \tag{22}$$

Condition (22) clearly holds if there is a function w satisfying the hypotheses of Theorem 2. In [6], g is assumed to satisfy (2), which implies (22), because $V_L(x) \geq g_o[\sup_\alpha t(x, \alpha)] \rightarrow +\infty$ as $x \rightarrow \partial(\mathcal{D}_o)$. More generally, (22) will hold if we add the assumption

(A_8) $\mathbb{R}^N$ is g quasi-stable

We say that g satisfies the **standing hypotheses** if (A_1)-(A_4) and (A_6)-(A_8) hold. We now show that (22) holds under the added hypothesis (A_8). Set

$$V_L^r(x) = \sup_{a \in \mathcal{A}^r} J[g](x, +\infty, \alpha) \ \in \ [0, +\infty]$$

We sometimes write $V_L^r[g]$ instead of V_L^r, to emphasize the cost function g. Recall (cf. [3]) that (17) has exactly the same set of solutions on any open set as

$$\inf_{a \in A^r} \{-f^r(x, a) \cdot Dv(x) - g^r(x, a)\} = 0, \tag{23}$$

and that V_L^r is a continuous solution of (23) on $\mathcal{D}_o^r$, by [6]. Recall that f^r is ULES. By Lemma 2.3, we know that $\mathcal{D}_o^r$ is open and that $\mathcal{D}_o^r = \mathcal{D}^r$, by the convexity of the sets $f^r(x, A^r)$. Notice that (A_8) and the equivalence of g quasi-stability and relaxed g quasi-stability (cf. Appendix A) imply that $\mathrm{dom}(V_L^r) \subseteq \mathcal{D}^r$. Therefore, $\mathrm{dom}(V_L^r) = \mathcal{D}_o^r$, since V_L^r is finite on $\mathcal{D}_o^r$ (cf. [6]). By Theorem 2, (22) will follow if

$$V_L^r(x) \rightarrow +\infty \ \text{ as } \ x \rightarrow x_o \ \text{ for all } \ x_o \in \partial\left(\mathcal{D}_o^r\right), \tag{24}$$

since then we can take $w = V_L^r$ and $\mathcal{O} = \mathcal{D}_o^r$ in Theorem 2 and conclude that $\mathcal{D}_o^r = \mathcal{D}_o$ and $V_L^r = V_L$ on $\mathcal{D}_o$. We will now prove (24).

Suppose that $x_n \rightarrow x_o \in \partial(\mathcal{D}_o^r)$, but that there exists K such that $V_L^r(x_n) \leq K < \infty$ for all n. Fix $a \in \mathcal{A}^r$ and $M > 0$. Set

$$\mathcal{S}_M = B_1(\{\phi(t, x_o, a) : 0 \leq t \leq M\}),$$

and choose $\bar{F} > 0$ such that $||f(x, a) - f(y, a)|| \leq \bar{F}||x - y||$ for all x and y in $B_1(\mathcal{S}_M)$ and all $a \in A$. There exists an $N_M \in \mathbb{N}$ such that

$$||\phi(t, x_o, a) - \phi(t, x_n, a)|| \ \leq \ e^{t\bar{F}}||x_n - x_o|| \quad \forall t \in [0, M], \ \ \forall n \geq N_M, \tag{25}$$

and such that $\phi(t, x_n, a) \in \mathcal{S}_M$ for all $t \in [0, M]$ and all $n \geq N_M$. This follows from Gronwall's Inequality and a generalization of standard estimates from [3]. By enlarging N_M, we can guarantee that

$$||x_n - x_o|| \ \leq \ \frac{1}{M\left[\bar{g} + 1\right]} e^{-M\bar{F}} \quad \forall n \geq N_M \tag{26}$$

where $|g(x,a) - g(y,a)| \leq \bar{g}||x - y||$ for all $x, y \in B_1(\mathcal{S}_M)$ and $a \in A$. Combining (25) and (26),

$$g^r(\phi(t, x_o, a), a(t)) \;\leq\; g^r(\phi(t, x_n, a), a(t)) \;+\; 1/M \quad \forall n \geq N_M \tag{27}$$

a.e. $t \in [0, M]$. We can integrate (27) to get

$$J[g](x_o, M, a) \leq K + 1 \;\; \forall a \in \mathcal{A}^r, M > 0. \tag{28}$$

Letting $M \to +\infty$ in (28) for fixed a, and using the equivalence of g quasi-stability and relaxed g quasi-stability, we conclude that $x_o \in \mathcal{D}^r$. Recalling that $f^r(x, A^r)$ is convex for all x, Lemma 2.3(c) gives $x_o \in \mathcal{D}_o^r$, which contradicts the fact that $\mathcal{D}_o^r$ is open. The following proposition summarizes these observations:

Corollary 4.1 Let (A_1)-(A_4) and (A_6)-(A_8) hold with f ULES. Then V_L is a robust Lyapunov function for f on $\mathcal{D}_o$ which satisfies (22). Moreover, $\mathcal{D}_o = \mathcal{D}_o^r$, and $V_L \equiv V_L^r$ on $\mathcal{D}_o$.

Remark 4.2 Under our standing hypotheses,

$$\mathcal{D}_o \subseteq \mathrm{dom}(V_L) \subseteq \mathcal{D} \subseteq \mathrm{cl}(\mathcal{D}_o),$$

which is weaker than the condition $\mathrm{dom}(V_L) = \mathcal{D}_o$. Later, we show that under (A_1)-(A_8), we do in fact have $\mathrm{dom}(V_L) = \mathcal{D}_o$. Our hypotheses are *weaker* than (2), since (A_7) allows vanishing and asymptotically decaying costs g (cf. [14], and Remark 4.9 and §5 below). For the study of the case where (2) holds, see [4, 5, 6, 8]. Notice that (A_1)-(A_8) imply that $\mathbb{R}^N$ is also relaxed g quasi-stable (cf. Appendix A). Also, the proof of (22) remains valid if (A_8) is relaxed to the requirement that there be a g quasi-stable set containing $\mathrm{cl}(\mathcal{D}_o)$.

Remark 4.3 Under the additional hypothesis (2), $V_L[g]$ is proper (cf. [6]). However, under the standing hypotheses on g, $V_L[g]$ may not be proper. For example, we can take

$$N = 1, \;\; g(x) = |x|/(1 + x^2), \;\; \text{and} \;\; f(x, a) \equiv -x,$$

in which case $V_L[g](x) \to \pi/2$ as $x \to +\infty$.

4.2.2 Lipschitz Lyapunov Functions

It is natural to look for conditions under which the Lyapunov function V_L is not only continuous on $\mathcal{D}_o$, but also locally Lipschitz on $\mathcal{D}_o$. It is also natural to ask whether the function

$$v_\delta := 1 - e^{-\delta V_L[g]}$$

is globally Lipschitz for large enough fixed $\delta > 0$, because $v_\delta(x) \to 1$ as $\delta \to +\infty$ for each fixed $x \in \mathcal{D}_o \setminus \{0\}$. Later, we will show that v_δ is a robust Lyapunov function for f on $\mathcal{D}_o$ for each fixed $\delta > 0$, and that it is a global solution of the Zubov equation for suitable g. The question of whether v_δ is globally Lipschitz for fixed $\delta > 0$ therefore reduces to the question of whether the methods of the previous section can produce globally Lipschitz Lyapunov functions and globally Lipschitz solutions of (1). The proof of the following is the same as the proofs of Proposition 4.2 and Proposition 4.3 in [6]:

Proposition 4.4 Let (A_1)-(A_4) and (A_6)-(A_7) hold with f ULES, and assume the following:

(L) There are $c, \kappa > 0$ and $s > L_r/\sigma$ such that for all $x, y \in B_c$ and $a \in A$,

$$|g(x,a) - g(y,a)| \leq \kappa \max\{||x||^s, ||y||^s\}||x-y||,$$

where r and σ are the constants in the ULES definition and L_r is the constant from (A_2).

Then V_L is locally Lipschitz on $\mathcal{D}_o$. If in addition there are $L_f \in (0, s\sigma)$ and $L_g > 0$ such that

$$||f(x,a) - f(y,a)|| \leq L_f||x-y|| \text{ and } |g(x,a) - g(y,a)| \leq L_g||x-y|| \quad (29)$$

for all $x, y \in \mathbb{R}^N$ and $a \in A$, and if (2) holds, then v_δ is globally Lipschitz on $\mathbb{R}^N$ for all sufficiently large $\delta > 0$.

The proof of the global Lipschitzness of v_δ is based on the fact that $V_L(x) > g_o[\sup\{t(x,a) : a \in \mathcal{A}\}]$, where g_o is the uniform lower bound for g in (2). The following example shows that (2) cannot be deleted from the hypotheses of this proposition. It illustrates how relaxing the uniform lower bound on g gives phenomena not found in the Zubov equation solutions in [6] (see also Remark 4.8).

Example 4.5 For each $k \in \mathbb{N}_o := \{0, 1, 2, \ldots\}$, set

$$t_{k-} = 10^k - \frac{1}{10^{2k+1}} \text{ and } t_{k+} = 10^k + \frac{1}{10^{2k+1}}.$$

Define I_Δ and $\Delta : I_\Delta \to \mathbb{R}$ by

$$I_\Delta = \bigcup_{k \in \mathbb{N}_o} [t_{k-}, t_{k+}], \; \Delta(x) = \begin{cases} 10^{3k+1}(x - t_{k-}), & t_{k-} \leq x \leq 10^k, \; k \in \mathbb{N}_o \\ 10^{3k+1}(t_{k+} - x), & 10^k \leq x \leq t_{k+}, \; k \in \mathbb{N}_o \end{cases}$$

Then the graph of Δ is a sequence of nonoverlapping triangles centered at the points 10^k for $k \in \mathbb{N}_o$ which become taller and thinner as $k \to +\infty$. In fact, while

$$\Delta(10^k) = 10^k$$

for all $k \in \mathbb{N}_o$, we have

$$\int_{I_\Delta} \Delta(x)\,\mathrm{d}x = \frac{1}{10}\sum_{k=0}^{\infty} 10^{-k} < \infty$$

Let $Q : \mathbb{R} \to \mathbb{R}$ be any continuous function that satisfies the following conditions.

- $Q \equiv \Delta$ on I_Δ, $Q(x) = (\frac{9}{10})^6 x^6$ for $0 \le x \le \frac{9}{20}$, and $Q(x) \ge 0$ for all $x \ge 0$
- $\mathrm{Null}(Q) = \{0, \pm t_{1-}, \pm t_{1+}, \pm t_{2-}, \pm t_{2+}, \ldots\}$, and $|Q(x)| \le 1$ for all x in $\mathrm{comp}(I_\Delta)$
- $Q\lceil[\mathrm{comp}\,(I_\Delta)]$ is Lipschitz with Lipschitz constant $\mathcal{L} \le 1$, Q is odd, and $\int_{[0,\infty)} Q(x)\,\mathrm{d}x < \infty$

We leave the easy construction of Q to the reader. The functions

$$g(x) = -Q(x)f(x), \qquad f(x) = -\begin{cases} \left(\frac{10}{9}\right)^6 x, & -\frac{9}{10} \le x \le \frac{9}{10} \\ \frac{1}{x^5}, & x \ge \frac{9}{10} \text{ or } x \le -\frac{9}{10} \end{cases}$$

satisfy (29) with $L_g > 0$ and $L_f \in (0, 6(\frac{10}{9})^6)$. Take f as the dynamics, with no controls, and g as the cost function. Then (A_1)-(A_4) and (A_6)-(A_7) hold with f ULES, and (L) holds with parameters $r = 9/10$, $C = 1$, $\sigma = L_r = (10/9)^6$, $s = 6$, $\kappa = 7$, and $c = 9/20$. We let $\phi(t, x)$ denote the trajectory for f and the initial value x. For all $x > 0$, it follows that

$$V_L(x) = \int_0^\infty g(\phi(t,x))\,\mathrm{d}t = \int_0^\infty \frac{g(\phi(t,x))}{f(\phi(t,x))}\frac{\partial\phi}{\partial t}(t,x)\,\mathrm{d}t = \int_0^x Q(u)\,\mathrm{d}u.$$

Therefore, if $\delta > 0$ is given, and if we set $v_\delta(x) = 1 - e^{-\delta V_L(x)}$, then we get

$$|Dv_\delta(10^k)| = \delta Q(10^k)\exp\left(-\delta\int_0^{10^k} Q(s)\,\mathrm{d}s\right) \to +\infty \;\text{ as }\; k \to +\infty.$$

Therefore, while V_L is locally Lipschitz, there cannot exist $\delta > 0$ such that v_δ is globally Lipschitz. This example shows that the positivity condition (2) cannot be omitted from the statement of Corollary 4.4.

4.3 Uniqueness of Viscosity Solutions of Generalized Zubov Equation

For any function $w : \mathbb{R}^N \to (-\infty, +\infty]$, we define the **Kružkov transformation** $\breve{w} : \mathbb{R}^N \to \mathbb{R}$ of w by $\breve{w}(x) := 1 - e^{-w(x)}$, with the convention that $e^{-\infty} = 0$. If w is a solution of the auxiliary PDE (17) on $\mathcal{D}_o$, then $\breve{w}$ is a solution of the generalized Zubov equation (1) on $\mathcal{D}_o$ (cf. [3], Chapter II).

We apply this observation to V_L, assuming for the rest of this subsection that the hypotheses of Corollary 4.1 are satisfied.

Recall that $\mathrm{dom}(V_L^r) = \mathcal{D}_o^r (= \mathcal{D}^r)$. Since $V_L^r(x) \to +\infty$ as $x \to \partial(\mathcal{D}_o^r)$ and V_L^r is continuous on $\mathcal{D}_o^r$, $\check{V}_L^r$ is continuous on $\mathbb{R}^N$. One can check (cf. [6]) that $\check{V}_L^r$ satisfies the Dynamic Programming Principle

$$\begin{aligned}&\check{V}_L^r(x) = \sup_{\alpha\in\mathcal{A}^r} \left\{[1 - G(x,t,\alpha)] + G(x,t,\alpha)\check{V}_L^r(\phi(t,x,\alpha))\right\} \quad \forall t \geq 0,\\ &G(x,t,\alpha) := e^{-J[g](x,t,\alpha)}\end{aligned} \tag{30}$$

on $\mathbb{R}^N$. Using (30), one can use the arguments of [3] to show that $\check{V}_L^r$ is a viscosity solution of the generalized Zubov equation (1) on all of $\mathbb{R}^N$. Also, (30) and (A_7) give

$$\check{V}_L(\phi(t,x,\alpha)) \quad \leq \quad [1 - G(x,t,\alpha)]\left[\check{V}_L(\phi(t,x,\alpha)) - 1\right] + \check{V}_L(x) \; < \; \check{V}_L(x)$$

for all $x \in \mathcal{D}_o \setminus \{0\}$, $\alpha \in \mathcal{A}$ and $t > 0$, since $V_L^r = V_L$ on $\mathcal{D}_o$ (cf. §4.2). Therefore, $\check{V}_L$ is also a robust Lyapunov function for f on $\mathcal{D}_o$.

Now let $w : \mathbb{R}^N \to \mathbb{R}$ be any solution of (1) on $\mathbb{R}^N$ that satisfies (SC_w). By applying Theorem 1 with $\ell \equiv -g$ and $h \equiv g$, we conclude that w agrees with

$$\check{V}_L(x) \quad = \quad \sup_{\alpha\in\mathcal{A}} \int_0^\infty G(x,t,\alpha)\, g(\phi(t,x,\alpha),\alpha(t))\,\mathrm{d}t \quad \in \mathbb{R} \tag{31}$$

on the asymptotically null set $\mathcal{D}_o$. In fact, if we also assume that w is bounded, then $w \equiv \check{V}_L$ on all of $\mathbb{R}^N$. To see why, we can assume w is continuous, the general case being proven in a similar way. First note that by the boundedness of w and (SC_w),

$$\lim_{t\to+\infty} G(x,t,\alpha)\, w(\phi(t,x,\alpha)) = 0 \quad \forall \alpha \in \mathcal{A},\ \forall x \in \mathbb{R}^N \tag{32}$$

which follows by separately considering the cases where the exponent

$$\int_0^\infty g(\phi(s,x,\alpha),\alpha(s))\,\mathrm{d}s$$

converges or diverges and using g quasi-stability. An application of the first part of Lemma 2.4, (SC_w), and (32) then gives $w(x) \geq J[g,g](x,+\infty,\alpha)$ for all $\alpha \in \mathcal{A}$ and $x \in \mathbb{R}^N$, so $w(x) \geq \check{V}_L(x)$. On the other hand, given $\varepsilon > 0$ and $x \in \mathbb{R}^N$, the construction in the proof of Theorem 1 gives an input $\hat{\alpha} \in \mathcal{A}$ such that

$$\begin{aligned}-w(x) \quad &\geq \quad -\int_0^t G(x,s,\hat{\alpha})\, g(\phi(s,x,\hat{\alpha}),\hat{\alpha}(s))\,\mathrm{d}s\\ &- \quad G(x,t,\hat{\alpha})\, w(\phi(t,x,\hat{\alpha})) - \varepsilon(1 - e^{-t})\end{aligned}$$

for all $t \in \mathbb{N}$. Combining this and (32) now gives

$$w(x) \; \leq \; J[g,g](x,+\infty,\hat{\alpha}) + \varepsilon \; \leq \; \check{V}_L(x) + \varepsilon,$$

as needed. Recall that $\check{V}_L^r$ is a continuous bounded viscosity solution of (1) on $\mathbb{R}^N$ which is null at the origin. Setting $w = \check{V}_L^r$ in the preceding argument therefore allows us to conclude that $\check{V}_L^r = \check{V}_L$ on all of $\mathbb{R}^N$, so $\check{V}_L$ is also a continuous solution on $\mathbb{R}^N$. The following corollary summarizes these observations and includes the global uniqueness results for the generalized Zubov equation in [6].

Corollary 4.6 Assume (A_1)-(A_4) and (A_6)-(A_8) with f ULES. Then:

(1) $\check{V}_L$ is a robust Lyapunov function for f on $\mathcal{D}_o$, and $\mathcal{D}_o = \check{V}_L^{-1}([0,1))$.

(2) If w is a solution of (1) on $\mathcal{D}_o$ satisfying (SC_w), then $w \equiv \check{V}_L$ on $\mathcal{D}_o$.

(3) If w is a bounded solution of (1) on $\mathbb{R}^N$ satisfying (SC_w), then $w \equiv \check{V}_L$ on $\mathbb{R}^N$.

Moreover, (i) $\check{V}_L$ is the unique bounded solution w of (1) on $\mathbb{R}^N$ that satisfies (SC_w) and (ii) $\mathcal{D}_o = w^{-1}([0,1))$ for any bounded solution w of (1) on $\mathbb{R}^N$ that satisfies (SC_w).

Remark 4.7 In Example 1.1, (A_1)-(A_4) and (A_6)-(A_7) hold, but (A_8) is not satisfied, and there are *infinitely many* bounded solutions of (1) on $\mathbb{R}^N$ satisfying (SC_w). Therefore, the quasi-stability hypothesis (A_8) of Corollary 4.6 cannot be dropped.

Remark 4.8 Conclusions (2)-(3) of the preceding corollary remain true if (A_6) is replaced by (19). We can also prove 'nonglobal' PDE characterizations for solutions of (1) on general open sets $\mathcal{O}$ under our relaxed conditions on g (cf. Appendix B). The paper [8] suggests the problem of determining what subset of the set of all robust Lyapunov functions V for a given ULES dynamics f has the following properties: (i) $V \equiv \check{V}_L[g]$ for some cost function g and (ii) V is the unique bounded solution of the generalized Zubov equation (1) that is null at the origin. One consequence of our results is that by allowing more general costs g, we made the subset of known Lyapunov functions satisfying these two properties strictly larger. In particular, the set of Lyapunov functions $\check{V}_L[g]$ studied in [6, 8] is a *proper* subset of the set of all functions that can be written as unique solutions of generalized Zubov equations. To see why, we have to find robust Lyapunov functions V which can be written as $\check{V}_L[g]$ for cost functions g satisfying our standing hypotheses, but which cannot be written as $\check{V}_L[\hat{g}]$ if $\hat{g}$ is also assumed to satisfy the positivity condition (2) which is assumed in [4, 5, 6, 8]. A general method for doing this is as follows.

Let f be a ULES dynamics satisfying (A_1)-(A_3) and $B_r \subsetneq \mathcal{D}_o$, and g satisfy the standing hypotheses in such a way that $g(x,a) \equiv \gamma(x)$ and $\gamma(\bar{x}) = 0$ for some $\bar{x} \in \mathcal{D}_o \setminus B_r$ (cf. §5 for particular cases). In particular, this means γ is locally Lipschitz (cf. (A_6)). Then g does not satisfy the positivity condition (2). Take W to be the associated Lyapunov function $V_L[g]$, so that $1 - e^{-W} = \check{V}_L[g]$. Then W is continuous on $\mathcal{D}_o$, and $\check{V}_L[g]$ is the unique bounded solution of (1) on $\mathbb{R}^N$ satisfying (SC_w). Suppose $\hat{g}$ satisfies the standing hypotheses and also the positivity condition (2), and suppose further that $W \equiv V_L[\hat{g}]$ on $\mathcal{D}_o$. Let $L_\gamma > 1$ be a Lipschitz constant for γ on $B_1(\bar{x})$, and pick $\varepsilon \in (0, g_o)$, where g_o is from the positivity condition (2) on $\hat{g}$. Since f is bounded, we can find $t > 0$ such that

$$||\phi(s,\bar{x},\alpha) - \bar{x}|| \leq \frac{\varepsilon}{L_\gamma} \wedge \left[\frac{1}{2}\text{dist}(\bar{x}, B_r \cup \partial(\mathcal{D}_o))\right] \wedge 1/2 \qquad \forall s \in [0,t],\ \forall \alpha \in \mathcal{A}^r$$

Using the Dynamic Programming Principle (21) for $W \equiv V_L[g]$, we can find a sequence $\alpha_n \in \mathcal{A}$ so that

$$W(\bar{x}) - W(\phi(t,\bar{x},\alpha_n)) \leq J[g](\bar{x},t,\alpha_n) + \frac{1}{n}.$$

A reapplication of Lemma 2.1 on the sequential compactness of $\mathcal{A}^r$, (21), and the fact that $W \equiv V_L[\hat{g}] \equiv V_L^r[\hat{g}] \equiv V_L[g] \equiv V_L^r[g]$ on $\mathcal{D}_o$ then give $\bar{\alpha} \in \mathcal{A}^r$ such that

$$\begin{aligned} g_o t &\leq \int_0^t \hat{g}^r(\phi(s,\bar{x},\bar{\alpha}), \bar{\alpha}(s))\,\mathrm{d}s \\ &\leq W(\bar{x}) - W(\phi(t,\bar{x},\bar{\alpha})) \\ &\leq \int_0^t \gamma^r(\phi(s,\bar{x},\bar{\alpha}))\,\mathrm{d}s \\ &\leq tL_\gamma \max\{||\phi(s,\bar{x},\bar{\alpha}) - \bar{x}|| : 0 \leq s \leq t\} \ \leq\ t\varepsilon, \end{aligned}$$

contradicting the choice of ε. This shows that $V := 1 - e^{-W}$ satisfies the requirement.

Remark 4.9 Notice that Corollary 4.6 applies to cases where (2) is not satisfied, and which therefore are not covered by [6]. For example, take any bounded ULES f and

$$g(x,a) \equiv \begin{cases} \frac{1}{1+r}\frac{||x||}{r}, & ||x|| \leq r \\ \frac{1}{1+||x||}, & ||x|| > r \end{cases}$$

where r is the constant in the ULES definition, in which case the g quasi-stability of $\mathbb{R}^N$ is easily shown. In this way, Corollary 4.6 generalizes the uniqueness results of [6] on (1) by establishing stronger conclusions that also cover a larger class of cost functions g. Moreover, Remark 4.8 shows that Corollary 4.6 gives PDE characterizations for a strictly larger class of Lyapunov functions than is covered by [6].

Remark 4.10 Recall that by the "$\mathcal{KL}$-Lemma" (cf. [19]), each $\beta_f \in \mathcal{KL}$ admits $\alpha_1, \alpha_2 \in \mathcal{K}^\infty$ such that

$$\beta_f(r,t) \ \leq\ \alpha_2(\alpha_1(r)e^{-t}) \quad \forall r,t \geq 0 \qquad (33)$$

The results we gave in this note remain true if we replace the assumption (19) with the condition that there be positive constants ε and δ such that

$$0 \leq g(x,a) \leq \delta\alpha_2^{-1}(||x||) \quad \forall (x,a) \in B_\varepsilon \times A \tag{34}$$

and relax the ULES assumption to the requirements that (i) f is ULAS and (ii) the requirement ($\star\star$) holds for $\beta_f \in \mathcal{KL}$ satisfying (33) (and keep the rest of the assumptions the same). Condition (34) is used in [4] to study the Zubov equation for ULAS dynamics, but under the extra conditions (2) and (A_6). (In [4], f is assumed to satisfy a more general ULAS-like condition, involving a general compact attraction set D instead of the origin. The arguments we gave above can be adapted to this slightly more general case.) Notice however that the results we gave on verification functions w in §3 (for $\ell = -g$) remain true for general asymptotically null dynamics, even if (34) is not assumed (cf. Remark 2.6).

5 Illustrations

This section illustrates our results by revisiting a simple example from [6], where the ULES dynamics is

$$f(x,a) = (-x_1 + ax_1^2, -x_2 + ax_2^2), \quad (x,a) \in \mathbb{R}^2 \times [-1,+1] \tag{35}$$

(We can take $f \equiv 0$ outside a suitable set containing $(-1,+1)^2 \times A$ to bound the dynamics.) If we choose $g(x) = ||x||^2$, then the positivity condition (2) on g holds, and [6] shows that

$$V_L(x) = \begin{cases} -\ln(1-x_1) - \ln(1-x_2) - x_1 - x_2, & x_1 \geq -x_2 \\ -\ln(1+x_1) - \ln(1+x_2) + x_1 + x_2, & x_1 \leq -x_2 \end{cases}$$

on $\mathcal{D} = \mathcal{D}_o = (-1,1)^2$. By [6], this function is the unique continuous *positive solution* of

$$x \cdot Dw(x) - |x_1^2(Dw(x))_1 + x_2^2(Dw(x))_2| - ||x||^2 = 0 \tag{36}$$

on $\mathcal{D}_o$ that satisfies

$$(SC_w), \;\; w \text{ is bounded-from-below, and } w(x) \to +\infty \text{ as } x \to \partial(\mathcal{D}_o) \tag{37}$$

On the other hand, Theorem 2 proves a stronger result, namely, that V_L is in fact the unique *solution* of (36) on $\mathcal{D}_o$ in the class of functions $w : \mathcal{D}_o \to \mathbb{R}$ that satisfy (37). In particular, there are no discontinuous solutions of (36) on $\mathcal{D}_o$ that satisfy (37).

Our results also apply to more general cases. For example, change g to

$$\hat{g}(x,a) \equiv ||x||^2\Psi(x), \text{ where } \Psi(x) = \sqrt{|x_1 - 3/4|} + \sqrt{|x_2 - 3/4|}. \tag{38}$$

By Corollary 4.1, and the fact that $\Psi(x) \geq |x_1 - \frac{3}{4}| + |x_2 - \frac{3}{4}|$ for all $x \in \mathcal{D}_o$ with $x_1, x_2 \geq 0$, one shows that the Lyapunov function $V_L[\hat{g}]$ still satisfies (22), and $V_L[\hat{g}]$ is still a robust Lyapunov function for f on $\mathcal{D}_o$ (cf. [3]). Also, $V_L[\hat{g}]$ is still a solution of

$$x \cdot Dw(x) - |x_1^2(Dw(x))_1 + x_2^2(Dw(x))_2| - ||x||^2\Psi(x) \;=\; 0 \qquad (39)$$

on $\mathcal{D}_o$, and it is the unique solution of this equation on $\mathcal{D}_o$ that satisfies (37), by Theorem 2. Since the Dynamic Programming Principle still holds on $\mathbb{R}^N$, $\check{V}_L[\hat{g}]$ is the unique bounded solution of the generalized Zubov equation

$$||x||^2\Psi(x)\,[w(x) - 1] + x \cdot Dw(x) - |x_1^2(Dw(x))_1 + x_2^2(Dw(x))_2| \;=\; 0 \quad (40)$$

on $\mathbb{R}^N$ that satisfies (SC_w) (cf. [3] and §4.3). Since the Lipschitz and positivity conditions on g are no longer satisfied, these results do not follow from the known results. The preceding results remain true if Ψ in (38) is replaced by

$$\Psi(x) = |x_1 - 3/4| + |x_2 - 3/4|.$$

More generally, for any $g : \mathbb{R}^2 \times [-1, +1] \to [0, \infty)$ satisfying the standing hypotheses, we get a maximal cost type robust Lyapunov function V_L for (35) and corresponding PDE characterizations for V_L, $\check{V}_L$, and $\mathcal{D}_o$.

6 Conclusion

This note analyzed a class of explicit robust Lyapunov functions of maximal cost type for uniformly locally asymptotically stable dynamics. These Lyapunov functions were shown to be unique viscosity solutions of first order equations, subject to appropriate side conditions. The Kružkov transformations of these Lyapunov functions are unique solutions of the generalized Zubov equation (1) introduced in [6]. Since we allowed general cost functions g, these uniqueness results do not follow from known results on first-order viscosity solutions. The uniqueness characterizations were used to give sublevel set characterizations for the robust domain of attraction. As a byproduct, we gave new PDE characterizations for the variable interest rate infinite horizon minimal cost function for cases where the Lagrangian could be negative, including the case where the discount rate is identically zero and the dynamics is unstable. This result is of independent interest, because it allows infinite horizon problems with unbounded cost functions where cost minimization can take place in one part of the state space while maximization takes place in the rest of the state space. One could consider the question of what subset of the set of all robust Lyapunov functions can be expressed as unique solutions of Zubov PDE's. As shown in Remark 4.8, our allowing general nonnegative g increased the size of this subset. One could also consider the question of how the numerical analysis of the generalized Zubov equation for positive cost functions g (cf. [5]) can be extended to the case of general nonnegative

g, and in particular, how allowing degenerate costs g affects the computation of the sublevel set $\mathcal{D}_o = \check{V}_L^{-1}([0,1))$. Research on these questions is ongoing.

A Appendix: Quasi-Stability and Existence of Nearly Optimal Trajectories

This appendix proves the following results on quasi-stability and nearly-optimal trajectories used in §§3-4:

Proposition A.1 Let (A_1)-(A_4) and (A_6) hold and $\mathcal{O} \subseteq \mathbb{R}^N$. Then $\mathcal{O}$ is relaxed g quasi-stable iff $\mathcal{O}$ is g quasi-stable.

Proposition A.2 Let (A_1), (A_2),(A_4), and (A_5) hold, with $h \equiv 0$ and $\text{Null}(||f(0,\cdot)||) \neq \emptyset$, $\mathcal{G} \subseteq \mathbb{R}^N$ be f-invariant, and $w : \mathcal{G} \to \mathbb{R}$ be a solution of (8) on $\mathcal{G} \setminus \{0\}$ satisfying (SC_w). Let $\bar{x} \in \mathcal{G}$, $i \in \mathbb{N}$, and $\varepsilon > 0$ be given, and define the functions E_j by (12). Then,

$$\mathcal{Z}_i := \left\{ \begin{array}{l} (t,\alpha) : \ 0 \le t \le 1, \ \alpha \in \mathcal{A}, \ w_\star(\bar{x}) \ge \int_0^t \ell(\phi(s,\bar{x},\alpha),\alpha(s))\,\mathrm{d}s \\ +w_\star(\phi(t,\bar{x},\alpha)) - E_i(t) \end{array} \right\} \quad (41)$$

contains an element of the form $(1,\bar{\alpha})$.

We start by proving Proposition A.1, which follows from the following special case of Theorem 1 in [10]:

Lemma A.3 Let A be a compact metric space, and let f be uniformly locally Lipschitz (cf. (A_2)). Let $x \in \mathbb{R}^N$, let $\alpha \in \mathcal{A}^r$ be such that $\phi(\cdot,x,\alpha)$ has domain $[0,\infty)$, and let $r : [0,\infty) \to (0,\infty)$ be continuous. Then there exist $\beta \in \mathcal{A}$ and $\eta^o \in B_{r(0)}(x)$ such that $||\phi(t,x,\alpha) - \phi(t,\eta^o,\beta)|| \le r(t)$ for all $t \ge 0$.

To prove Proposition A.1, we adapt the idea from [10] of putting an approximating trajectory in a tube around the reference trajectory which has a vanishing radius, along with an augmentation of the dynamics used in [7]. Extra care is taken to make sure that not only the reference trajectory is approximated, but also the *integrated cost* of the trajectory is approximated. The details are as follows. Let $\mathcal{O} \subseteq \mathbb{R}^N$ be open. If $\mathcal{O}$ is relaxed g quasi-stable, then it is also g quasi-stable, since as we remarked above, $\mathcal{A} \subseteq \mathcal{A}^r$. Conversely, assume $\mathcal{O}$ is g quasi-stable and $x \in \mathcal{O}$. Let $\alpha \in \mathcal{A}^r$ be such that $J[g](x,+\infty,\alpha) < +\infty$. The proposition will follow once we show that $\phi(t,x,\alpha) \to 0$ as $t \to +\infty$. We apply Lemma A.3 with the choices

$$r(t) := \left(\frac{1 \wedge \text{dist}(x,\partial\mathcal{O})}{2}\right) e^{-t}$$

and the *augmented dynamics*

$$\begin{cases} \dot{x}(t) = f(x(t), a(t)) \\ \dot{y}(t) = g(x(t), a(t)) \end{cases} , \quad a \in \mathcal{A} \tag{42}$$

Notice that the trajectory for (42), $a \in \mathcal{A}^r$, and any initial position $(\bar{x}, \bar{y})$ has the form

$$\left(\phi(t, \bar{x}, a),\ \bar{y} + \int_0^t g^r(\phi(s, \bar{x}, a), a(s))\,\mathrm{d}s\right)$$

Using the initial value $(x, 0) \in \mathbb{R}^{N+1}$, Lemma A.3 gives an input $\beta \in \mathcal{A}$ and an initial value $\eta^o = (\eta_L^o, \eta_R^o) \in \mathbb{R}^{N+1}$ satisfying

$$\begin{aligned} ||\phi(t, \eta_L^o, \beta) - \phi(t, x, \alpha)||^2 \quad &+ \quad \left|\eta_R^o + \int_0^t g(\phi(s, \eta_L^o, \beta), \beta(s))\,\mathrm{d}s \right. \\ &- \quad \left. \int_0^t g^r(\phi(s, x, \alpha), \alpha(s))\,\mathrm{d}s\right|^2 \\ &\leq \quad r^2(t) \ \ \forall t \geq 0 \end{aligned} \tag{43}$$

In particular, $||\eta_L^o - x|| \leq \mathrm{dist}(x, \partial\mathcal{O})/2$, so $\eta_L^o \in \mathcal{O}$. Since g is nonnegative, (43) gives

$$\textstyle\int_0^\infty g(\phi(t, \eta_L^o, \beta), \beta(t))\,\mathrm{d}t \ \ \leq \ \ |\eta_R^o| + \int_0^\infty g^r(\phi(t, x, \alpha), \alpha(t))\,\mathrm{d}t \ < \ \infty\,.$$

Since $\mathcal{O}$ is g quasi-stable, we get

$$\phi(t, \eta_L^o, \beta) \to 0 \ \text{ as } \ t \to +\infty,$$

so (43) gives

$$\begin{aligned} ||\phi(t, x, \alpha)|| \quad &\leq \quad ||\phi(t, x, \alpha) - \phi(t, \eta_L^o, \beta)|| + ||\phi(t, \eta_L^o, \beta)|| \\ &\leq \quad e^{-t} + ||\phi(t, \eta_L^o, \beta)|| \ \to \ 0 \ \text{ as } \ t \to \infty, \end{aligned}$$

as needed.

We turn next to the proof of Proposition A.2, which is a generalization of the proof of the theorem of [17]. We assume w is continuous and $i = 1$, the general case being similar. (See Theorem 2 for the justification for this continuity assumption.) Note that $\mathcal{Z}_1$ is partially ordered by

$$(t_1, \alpha_1) \sim (t_2, \alpha_2) \ \text{ iff } \ [t_1 \leq t_2 \text{ and } \alpha_2\lceil[0, t_1] \equiv \alpha_1 \text{ a.e.}] \tag{44}$$

Since $\mathcal{G}$ is f-invariant, the argument from [17] shows that every totally ordered subset of $\mathcal{Z}_1$ has an upper bound in $\mathcal{Z}_1$. (If $\{(t_j, \alpha_j)\}$ is totally ordered in $\mathcal{Z}_1$, then its upper bound in $\mathcal{Z}_1$ is $(\tilde{t}, \tilde{\alpha})$, where $\tilde{t} = \sup_j t_j$ and $\tilde{\alpha}(t) := \alpha_j(t)$ for a.e. $t \in [0, t_j]$.) It follows from Zorn's Lemma that $\mathcal{Z}_1$ contains a maximal element $(\bar{t}, \bar{\alpha})$. We will now show that $\bar{t} = 1$. We can assume that $\phi(\bar{t}, \bar{x}, \bar{\alpha}) \neq 0$ (since $\ell(0, a) \equiv w(0) = 0$ and $f(0, \bar{a}) = 0$ for some $\bar{a} \in A$).

Since $\mathcal{G}$ is f-invariant, $\phi(\bar{t}, \bar{x}, \bar{\alpha}) \in \mathcal{G}$. Let B be an open set containing $\phi(\bar{t}, \bar{x}, \bar{\alpha})$ whose closure lies in $\mathcal{G} \setminus \{0\}$. Suppose that $\bar{t} < 1$, set $q := \phi(\bar{t}, \bar{x}, \bar{\alpha})$, and pick $\delta \in [0, \mathrm{dist}(q, \partial B)/2)$. Since f is bounded, it follows that $T_\delta(q)$, as defined in (9), is positive or $+\infty$. By the second part of Lemma 2.4, it follows that there is a $t \in (0, 1 - \bar{t}\,)$ and a $\beta \in \mathcal{A}$ so that

$$\begin{aligned} w(\phi(\bar{t}, \bar{x}, \bar{\alpha})) \quad &\geq \quad \int_0^t \ell(\phi(s, \phi(\bar{t}, \bar{x}, \bar{\alpha}), \beta), \beta(s))\,\mathrm{d}s \\ &+ \quad w(\phi(t, \phi(\bar{t}, \bar{x}, \bar{\alpha}), \beta)) \\ &- \quad E_1(\bar{t} + t) \; + \; E_1(\bar{t}\,) \end{aligned} \tag{45}$$

and so that $\phi(s, \phi(\bar{t}, \bar{x}, \bar{\alpha}), \beta) \in B$ for all $s \in [0, t]$. Let $\beta^\sharp$ denote the concatenation of $\bar{\alpha}\lceil[0, \bar{t}\,]$ followed by the input β. If we now combine (45) with the inequality in (41) with the choice $\alpha = \bar{\alpha}$, then we get

$$w(\bar{x}) \;\; \geq \;\; \int_0^{\bar{t}+t} \ell(\phi(s, \bar{x}, \beta^\sharp), \beta^\sharp(s))\,\mathrm{d}s \; + \; w(\phi(\bar{t}+t, \bar{x}, \beta^\sharp)) \; - \; E_1(\bar{t}+t). \tag{46}$$

Since t was chosen so that $\bar{t} + t < 1$, we conclude from (46) that

$$\big(\bar{t} + t, \beta^\sharp\big) \in \mathcal{Z}_1.$$

Since $\beta^\sharp$ is an extension of $\bar{\alpha}$, this contradicts the maximality of the pair $(\bar{t}, \bar{\alpha})$. Therefore, $\bar{t} = 1$. This proves Proposition A.2 and completes the proof of Theorem 1 for $h \equiv 0$. The proof for general h is similar.

Remark A.4 It should be emphasized that Lemma A.3 is *not* an extension of the Filippov-Ważewski Relaxation Theorem, since the original and approximating trajectories are allowed to have different starting values. Theorem 1 in [10] is stated in terms of time-dependent differential inclusions $\dot{x} \in F(t, x)$. To get Lemma A.3 from this theorem, first take $F(t, x) \equiv f(x, A)$, so

$$\overline{\mathrm{co}}(F(t, x)) \equiv f^r(x, A^r).$$

The hypotheses of the theorem are satisfied since we are assuming that f is locally Lipschitz and A is bounded. The theorem allows us to approximate any solution x of $\dot{x} \in \overline{\mathrm{co}}(F(t, x))$ by a solution y of $\dot{y} \in F(t, y)$ in such a way that $||x(t) - y(t)|| \leq r(t)$ for all $t \geq 0$. Now take $\alpha \in \mathcal{A}^r$ and apply the preceding to the solution $x(t) := \phi(t, x, \alpha)$ of $\dot{x} \in \overline{\mathrm{co}}(F(t, x))$. Since the sets $A(t) := \{a \in A : f(y(t), a) = \dot{y}(t)\}$ are measurable, Filippov's Lemma gives an input $\beta \in \mathcal{A}$ such that $y(\cdot) = \phi(\cdot, \eta^o, \beta)$, where $\eta_o \in B_{r(0)}(x)$, which gives Lemma A.3.

B Appendix: Nonglobal Solutions of Zubov Equation

In §4, we gave global uniqueness characterizations for solutions of the PDE (1) on $\mathbb{R}^N$. As pointed out in [6], it can be inconvenient from a practical point of view to verify that a function is a PDE solution on all of $\mathbb{R}^N$. This motivates the problem of finding *non*global uniqueness characterizations for solutions of (1), on general open sets $\mathcal{O} \subseteq \mathbb{R}^N$. For simplicity, we will assume $\mathcal{D}_o \subseteq \mathcal{O}$. The following nonglobal uniqueness characterization extends the results of [6], §3, to general continuous g. The proof is a localization of the argument used to prove Theorem 1. Recall that if $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous, then a locally bounded function $w : \mathcal{O} \to \mathbb{R}$ is called a **(viscosity) supsersolution** (resp., **subsolution**) of

$$F(x, w(x), Dw(x)) = 0$$

on $\mathcal{O}$ provided that (C_1) (resp., (C_2)) of Definition 2.2 is satisfied. We will use the fact (cf. [3]) that if (A_1)-(A_5) hold, then any subsolution (resp., supersolution) of (8) on a bounded open set B satisfies conclusion (a) (resp., (b)) of Lemma 2.4.

Proposition B.1 Let (A_1)-(A_4) and (A_6)-(A_8) hold with f ULES. Assume that $\mathcal{O} \subseteq \mathbb{R}^N$ is open, $\mathcal{D}_o \subseteq \mathcal{O}$, and $w : \mathrm{cl}(\mathcal{O}) \to \mathbb{R}$ is bounded.

(i) If w is an upper semicontinous function which is a subsolution of (1) on $\mathcal{O}$ satisfying $w(0) \leq 0$ and $w \equiv 1$ on $\partial(\mathcal{O})$, then $w \leq \check{V}_L[g]$ on $\mathcal{O}$.

(ii) If w is a lower semicontinous function which is a supersolution of (1) on $\mathcal{O}$ satisfying $w(0) \geq 0$ and $w \geq 1$ on $\partial(\mathcal{O})$, then $w \geq \check{V}_L[g]$ on $\mathcal{O}$.

Proof. (i) By the proof of Theorem 1 (with $\ell \equiv -g$ and $h \equiv g$), $w \leq \check{V}_L[g]$ on $\mathcal{D}_o$. Let $\bar{x} \in \mathcal{O} \setminus \mathcal{D}_o$. It remains to show that

$$w(\bar{x}) \leq \check{V}_L[g](\bar{x}).$$

Assume the contrary and pick $\varepsilon > 0$ such that

$$w(\bar{x}) \geq \check{V}_L[g](\bar{x}) + \varepsilon \tag{47}$$

Define the functions E_i by (12) for $i \in \mathbb{N}$. Set $\bar{\phi}(0) = \bar{x}$, and

$$J_w(x, t, \alpha) = G(x, t, \alpha) - 1 - G(x, t, \alpha) w(\phi(t, x, \alpha))$$

wherever the RHS is defined, where G is as defined by (30). We will inductively define the sets

$$\mathcal{Z}_{\mathcal{O},i+1} := \left\{ \begin{array}{l} (t, \alpha) \in [0,1] \times \mathcal{A} : \quad \phi(s, \bar{\phi}(i), \alpha) \in \mathcal{O} \ \ \forall s \in [0, t], \\ -w(x) \ \geq \ J_w(\bar{\phi}(i), t, \alpha) - \frac{1}{2} E_{i+1}(t) \end{array} \right\}$$

for all $i \in \mathbb{N}_o$. The set $\mathcal{Z}_{\mathcal{O},1}$ is partially ordered by $\sim$ as defined in (44). Let

$$\hat{T} := \{(t_j, \alpha_j)\}$$

be a totally ordered subset of $\mathcal{Z}_{\mathcal{O},1}$. If $\phi(t_j, \bar{x}, \alpha_j)$ converges to point in $\partial(\mathcal{O})$, then the semicontinuity of w implies that $-\check{V}_L[g](\bar{x}) - \varepsilon$ majorizes

$$\begin{aligned} -w(\bar{x}) &\geq G(\bar{x}, t_j, \alpha_j) - 1 - G(\bar{x}, t_j, \alpha_j)\, w(\phi(t_j, \bar{x}, \alpha_j)) - \frac{\varepsilon}{2} \\ &\geq G(\bar{x}, t_j, \alpha_j) - 1 - G(\bar{x}, t_j, \alpha_j)\left(1 + \frac{\varepsilon}{4}\right) - \frac{\varepsilon}{2} \\ &\geq -1 - \frac{3\varepsilon}{4} \end{aligned} \tag{48}$$

for large j, so $\check{V}_L[g](\bar{x}) \leq 1 - \varepsilon/4$. Therefore, $\bar{x} \in \mathrm{dom}(V_L[g]) = \mathcal{D}_o$. This contradicts the choice of $\bar{x}$. By the proof of Proposition A.2, it follows that $\hat{T}$ has an upper bound in $\mathcal{Z}_{\mathcal{O},1}$ (cf. [17]). By Zorn's Lemma, $\mathcal{Z}_{\mathcal{O},1}$ contains a maximal element $(\bar{t}, \bar{\alpha})$, and the proof of Theorem 1 (with $\ell = -g$ and $g \equiv h$, applied to the supersolution $-w$ of the Hamilton-Jacobi equation (8)) shows that $\bar{t} = 1$. Now set

$$\bar{\phi}(1) = \phi(1, \bar{x}, \bar{\alpha}).$$

The preceding argument gives a maximal element $(1, \bar{\alpha}_2) \in \mathcal{Z}_{\mathcal{O},2}$. (To show that each totally ordered set of $(s_j, \alpha_{j,2})$'s in $\mathcal{Z}_{\mathcal{O},2}$ has an upper bound in $\mathcal{Z}_{\mathcal{O},2}$, notice that if $\phi(s_j, \bar{\phi}(1), \alpha_{j,2}) \to p \in \partial(\mathcal{O})$ for some $p \in \mathbb{R}^N$, then (48) with t_j and α_j replaced by $1 + s_j$ and the concatenation of $\bar{\alpha}$ followed by $\alpha_{j,2}$, respectively, gives the same contradiction as before.) Set

$$\bar{\phi}(2) = \phi(1, \bar{\phi}(1), \bar{\alpha}_2).$$

This procedure is iterated exactly as in the proof of Theorem 1 and gives an input $\tilde{\alpha} \in \mathcal{A}$ such that

$$w(\bar{x}) \;\leq\; 1 - G(\bar{x}, M, \tilde{\alpha}) + G(\bar{x}, M, \tilde{\alpha})\, w(\phi(M, \bar{x}, \tilde{\alpha})) + \frac{\varepsilon}{2} \;\; \forall M \in \mathbb{N}$$

and such that $\phi(t, \bar{x}, \hat{\alpha}) \in \mathcal{O}$ for all $t \geq 0$. A reapplication of (32) gives

$$w(\bar{x}) \;\leq\; 1 - G(\bar{x}, +\infty, \tilde{\alpha}) + \frac{\varepsilon}{2} \;\leq\; \check{V}_L[g](\bar{x}) + \frac{\varepsilon}{2},$$

which again contradicts the choice of $\varepsilon > 0$ in (47). Therefore,

$$w(\bar{x}) \leq \check{V}_L[g](\bar{x}),$$

which proves (i).
(ii) Let $\bar{x} \in \mathcal{O}$. For any $\alpha \in \mathcal{A}$, define the exit times

$$\tilde{t}(x, \alpha) := \inf\{t \geq 0 : \phi(t, x, \alpha) \notin \mathcal{O}\} \in [0, +\infty]$$

A repeated application of the first part of Lemma 2.4 with $\ell = -g$ (cf. [14]) gives

$$w(\bar{x}) \;\geq\; 1 - G(\bar{x}, t, \alpha) + G(\bar{x}, t, \alpha)\, w(\phi(t, \bar{x}, \alpha)) \;\; \forall \alpha \in \mathcal{A} \tag{49}$$

for all finite $t \in (0, \tilde{t}(\bar{x}, \alpha)]$. If $\tilde{t}(\bar{x}, \alpha)$ is finite for some $\alpha \in \mathcal{A}$, then (49) with the choice $t = \tilde{t}(\bar{x}, \alpha)$ gives

$$w(\bar{x}) \geq 1 \geq \check{V}_L[g](\bar{x}).$$

Otherwise, (32) and a passage to the liminf as $t \to +\infty$ in (49) for fixed α gives

$$w(\bar{x}) \geq 1 - G(\bar{x}, +\infty, \alpha)$$

for all $\alpha \in \mathcal{A}$, so $w(\bar{x}) \geq \check{V}_L[g](\bar{x})$, as needed. ∎

Remark B.2 If we put (i)-(ii) of the previous proposition together, then we get nonglobal PDE characterizations for $\check{V}_L[g]$ which extend the results of [6].

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