FOCK SPACES CORRESPONDING TO POSITIVE DEFINITE LINEAR TRANSFORMATIONS

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Abstract. Suppose \( A \) is a positive real linear transformation on a finite dimensional complex inner product space \( V \). The reproducing kernel for the Fock space of square integrable holomorphic functions on \( V \) relative to the Gaussian measure \( d\mu_A(z) = \sqrt{\det A} \cdot \text{Re}(A(z \, z)) \, dz \) is described in terms of the holomorphic–antiholomorphic decomposition of the linear operator \( A \). Moreover, if \( A \) commutes with a conjugation on \( V \), then a restriction mapping to the real vectors in \( V \) is polarized to obtain a Segal–Bargmann transform, which we also study in the Gaussian-measure setting.

INTRODUCTION

The classical Segal–Bargmann transform is an integral transform which defines a unitary isomorphism of \( L^2(\mathbb{R}^n) \) onto the Hilbert space \( F(\mathbb{C}^n) \) of entire functions on \( \mathbb{C}^n \) which are square integrable with respect to the Gaussian measure \( \mu = \pi^{-n}e^{-|z|^2} \, dx \, dy \), where \( dx \, dy \) stands for the Lebesgue measure on \( \mathbb{R}^{2n} \simeq \mathbb{C}^n \), see [1, 3, 4, 5, 10, 11]. There have been several generalizations of this transform, based on the heat equation or the representation theory of Lie groups [6, 9, 12]. In particular, it was shown in [9] that the Segal–Bargmann transform is a special case of the restriction principle, i.e., construction of unitary isomorphisms based on the polarization of a restriction map. This principle was first introduced in [9], see also [8], where several examples were explained from that point of view, In short the restriction principle can be explained in the following way. Let \( M_C \) be a complex manifold and let \( M \subset M_C \) be a totally real submanifold. Let \( F = F(M_C) \) be a Hilbert space of holomorphic functions on \( M_C \) such that the evaluation maps \( F \ni F \mapsto F(z) \in \mathbb{C} \) are continuous for all \( z \in M_C \), i.e., \( F \) is a reproducing Hilbert space. There exists a function \( K : M_C \times M_C \to \mathbb{C} \) holomorphic in the first variable, anti-holomorphic in the second variable, and such that the following hold:

(a) \( K(z, w) = \overline{K(w, z)} \) for all \( z, w \in M_C \);
(b) If \( K_w(z) := K(z, w) \) then \( K_w \in F \) and

\[ F(w) = (F, K_w), \quad \forall F \in F, z \in M_C. \]

The function \( K \) is the reproducing kernel for the Hilbert space, Let \( D : M \to \mathbb{C}^* \) be measurable. Then the restriction map \( RF := DF \mid_M \) is injective. Assume that there is a measure \( \mu \) on \( M \) such that \( RF \in L^2(M, \mu) \) for all \( F \) in a dense subset of

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F. Provided $R$ is closeable, polarizing $R^*$ we can write

$$R^* = U|R^*|$$

where $U : L^2(M, \mu) \to F$ is a unitary isomorphism. Using that $F$ is a reproducing Hilbert space we get that

$$U f(z) = (U f, K_z) = (f, U^* K_z) = \int_M f(m)(U^* K_z)(m) \, d\mu(m).$$

Thus $U f$ is always an integral operator. We notice also that the formula for $U$ shows that the important object in this analysis is the reproducing kernel $K(z, w)$. The reproducing kernel for the classical Fock space is given by $K(z, w) = e^{z\bar{w}}$. By taking $D(x) := (2\pi)^{-n/4}e^{-|x|^2}$, which is closely related to the heat kernel, we arrive at the classical Segal–Bargmann transform

$$U g(x) = (2/\pi)^{n/4} e^{(x,x)/2} \int g(y) e^{-(x-y,y-x)} \, dy.$$ 

The same principle can be used to construct the Hall–transform for compact Lie groups, [6]. In [2], Driver and Hall, motivated by application to quantum Yang–Mills theory, introduced a Fock space and Segal–Bargmann transform depending on two parameters $r, s > 0$, giving different weights to the $x$ and $y$ directions, where $z = x + iy \in \mathbb{C}^n$ (this was also studied in [12]). Thus $F$ is now the space of holomorphic functions $F(z)$ on $\mathbb{C}^n$ which are square-integrable with respect to the Gaussian measure $dM_{r,s}(z) = \frac{1}{(\pi)^{n/2}(2\pi)^{n/4}} e^{-\frac{x^2}{r} - \frac{y^2}{s}}$. In [12] the reproducing kernel and the Segal–Bargmann transform for this space is worked out. This construction has a natural generalization by viewing $r^{-1}$ and $s^{-1}$ as the diagonal elements in a positive definite matrix $A = d(r^{-1}I_n, s^{-1}I_n)$. The measure is then simply

(0.1) $$dM_{r,s}(z) = \sqrt{\det(A)} \frac{1}{\pi^n} e^{-(Az, z)} \, dx \, dy$$

and this has meaning for any positive definite matrix $A$.

In this paper we show that (0.1) gives rise to a Fock space $F_A$ for arbitrary positive matrices $A$. We find an expression for the reproducing kernel $K_A(z, w)$. We use the restriction principle to construct a natural generalization of the Segal–Bargmann transform for this space, with a certain natural restriction on $A$. We study this also in the Gaussian setting, and indicate a generalization to infinite dimensions.

1. **The Fock Space and the Restriction Principle**

In this section we recall some standard facts about the classical Fock space of holomorphic functions on $\mathbb{C}^n$. We refer to [5] for details and further information. Let $\mu$ be the measure $d\mu = \pi^{-n} e^{-\|z\|^2} \, dx \, dy$ and let $F$ be the classical Fock-space of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$||F||^2 := \int |F(z)|^2 \, d\mu(z) < \infty.$$ (Note that the term “Fock space” is also used for the completed symmetric tensor algebra over a Hilbert space, but that is not our usage here.) The space $F$ is a reproducing Hilbert space with inner product

$$(F, G) = \int F(z)\overline{G(z)} \, d\mu$$
and reproducing kernel $K(z, w) = e^{(z, w)}$, where $(z, w) = z \overline{w} = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n$. Thus

$$F(w) = \int F(z) \overline{K(z, w)} \, d\mu = (F, K_w)$$

where $K_w(z) = K(z, w)$. The function $K(z, w)$ is holomorphic in the first variable, anti-holomorphic in the second variable, and $K(z, w) = \overline{K(w, z)}$. Notice that $K(z, z) = (K_z, K_z)$. Hence $||K_z|| = e^{1/2}$. Finally the linear space of finite linear combinations $\sum c_j K_j, z_j \in \mathbb{C}^n, c_j \in \mathbb{C}$, is dense in $\mathbf{F}$. An orthonormal system in $\mathbf{F}$ is given by the monomials $e_\alpha(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}/\sqrt{\alpha_1! \cdots \alpha_n!}, \alpha \in \mathbb{N}_0^n$.

View $\mathbb{R}^n \subset \mathbb{C}^n$ as a totally real submanifold of $\mathbb{C}^n$. We will now recall the construction of the classical Segal-Bargmann transform using the restriction principle, see [8, 9]. For constructing a restriction map as explained in the introduction we need to choose the function $D(x)$. One motivation for the choice of $D$ is the heat kernel, but another one, more closely related to representation theory, is that the restriction map should commute with the action of $\mathbb{R}^n$ on the Fock space and $L^2(\mathbb{R}^n)$. Indeed, take

$$T(x)F(z) = m(x, z)F(z - x)$$

for $F$ in $\mathbf{F}$ where $m(x, z)$ has properties sufficient to make $x \mapsto T(x)$ a unitary representation of $\mathbb{R}^n$ on $\mathbf{F}$. Namely, $m$ is a multiplier, i.e., $m(x, z)m(y, z - x) = m(x + y, z); z \mapsto m(x, z)$ is holomorphic in $z$ for each $x$; and $|m(x, z)| = \left( \frac{dy(z - x)}{dy(z)} \right)^{1/2} = e^{(1/2)\|x\|^2}$. Note $m(x, z) := e^{x - \|x\|^2} \frac{1}{2}$ has these properties. Set $D(x) = (2\pi)^{-n/4}m(0, x) = (2\pi)^{-n/4}e^{-\|x\|^2/2}$ and define $R : \mathbf{F} \to C^\infty(\mathbb{R}^n)$ by

$$RF(x) := D(x)F(x) = (2\pi)^{-n/4}e^{-\|x\|^2/2}F(x).$$

Then

$$RT(y)F(x) = (2\pi)^{-n/4}e^{-\|x\|^2/2}T(y)F(x)$$

$$= (2\pi)^{-n/4}e^{-\|x\|^2/2}e^{x - y - \|y\|^2/2}F(x - y)$$

$$= (2\pi)^{-n/4}e^{-\|x - y\|^2/2}F(x - y)$$

$$= RF(x - y).$$

As $\mathbb{R}^n$ is a totally real submanifold of $\mathbb{C}^n$, it follows that $R$ is injective. Furthermore the holomorphic polynomials $p(z) = \sum a_\alpha z^\alpha$ are dense in $\mathbf{F}$ and obviously $Rp \in L^2(\mathbb{R}^n)$. Hence all the Hermite functions $h_\alpha(x) = (-1)^{\|\alpha\|} \left( D^\alpha e^{-\|x\|^2} \right) e^{\|x\|^2/2}$ are in the image of $R$; so $\text{Im}(R)$ is dense in $L^2(\mathbb{R}^n)$ and $R$ is a densely defined operator from $\mathbf{F}$ into $L^2(\mathbb{R}^n)$. It follows easily from the fact that the maps $F \mapsto F(z)$ are continuous, that $R$ is a closed operator. Hence $R$ has an adjoint $R^* : L^2(\mathbb{R}^n) \to \mathbf{F}$. For $z, w \in \mathbb{C}^n$, let $(z, w) = \sum z_j w_j$. Then:

$$R^*g(z) = (R^*g, K_z) = (g, RK_z)$$

$$= (2\pi)^{-n/4} \int g(y) e^{-\|y\|^2/2} e^{z \cdot y} \, dy$$

$$= (2\pi)^{-n/4} e^{(z, z)/2} \int g(y) e^{-(y - z, z - y)/2} \, dy$$

$$= (2\pi)^{n/4} e^{(z, z)/2} g \ast p(z)$$
where \( p(z) = (2\pi)^{-n/2}e^{-|z|^2/2} \) is holomorphic. Hence
\[
R R^* g(x) = g * p(x).
\]
(1.1)
As \( p \in L^1(\mathbb{R}^n) \), it follows that \( \|R R^*\| \leq |p|_1 \); so \( R R^* \) is continuous.
\[
(R^* g, R^* g) = (R R^* g, g) \leq \|R R^*\| \|g\|_2.
\]
Thus

**Lemma 1.1.** The maps \( R \) and \( R^* \) are continuous.

Let \( p_t(x) = (2\pi t)^{-n/2}e^{-|x_t|^2/2t} \) be the heat kernel on \( \mathbb{R}^n \). Then \( (p_t)_t \geq 0 \) is a convolution semigroup and \( p = p_1 \). Hence \( \sqrt{R R^*} = p_{1/2} \) or
\[
R U g(x) = |R^*| g(x) = p_{1/2} * g(x) = \pi^{-n/2} \int g(y)e^{-|x-y|^2} dy.
\]
It follows that
\[
U g(x) = (2/\pi)^{n/4} \int g(y)e^{-|z-y|^2} dy
\]
for \( x \in \mathbb{R}^n \). But the function on the right hand side is holomorphic in \( x \). Analytic continuation gives the following theorem.

**Theorem 1.2.** The map \( U : L^2(\mathbb{R}^n) \to F \) given by
\[
U g(z) = (2/\pi)^{n/4} \int g(y) \exp(-|y|^2 + 2y(z)) dy
\]
is a unitary isomorphism. \( U \) is called the Segal–Bargmann transform.

2. **Twisted Fock Spaces**

Let \( V \cong \mathbb{C}^n \) be a finite dimensional complex vector space of complex dimension \( n \) and let \( \langle \cdot, \cdot \rangle \) be a complex inner product. As before we will sometimes write \( \langle z, w \rangle = z \cdot w \). We will also consider \( V \) as a real vector space with real inner product defined by \( \langle z, w \rangle = \text{Re}(z, w) \). Notice that \( \langle z, z \rangle = \langle z, z \rangle \) for all \( z \in \mathbb{C}^n \). Let \( J \) be the real linear transformation of \( V \) given by \( Jz = iz \). Note that \( J^* = -J = J^{-1} \) and thus \( J \) is a skew symmetric real linear transformation. Fix a real linear transformation \( A \). Then \( A = H + K \) where
\[
H := \frac{A + J^{-1}AJ}{2} \quad \text{and} \quad K := \frac{A - J^{-1}AJ}{2}.
\]
Note that \( HJ = \frac{1}{2}(AJ - J^{-1}A) = \frac{1}{2}J(J^{-1}AJ + A) = JH \) and \( KJ = \frac{1}{2}(AJ + J^{-1}A) = \frac{1}{2}J(J^{-1}AJ - A) = -JK \). Furthermore \( H \) is complex linear and \( K \) is conjugate linear. We assume that \( A \) is symmetric and positive definite.

**Lemma 2.1.** The complex linear transformation \( H \) is self adjoint, positive with respect to the inner product \( \langle \cdot, \cdot \rangle \), and invertible.

**Proof.** Since \( A \) is positive and invertible as a real linear transformation, we have \( (Az, z) > 0 \) for all \( z \neq 0 \). But \( J \) is real linear and skew symmetric. Hence \( (JAJ^{-1}z, z) > 0 \) for all \( z \neq 0 \). In particular \( H = \frac{1}{2}(A + JAJ^{-1}) \) is complex linear, symmetric with respect to the real inner product \( \langle \cdot, \cdot \rangle \), and positive. We know \( (Hv, w) = (v, Hw) \). Thus \( \text{Re}(Hv, w) = \text{Re}(v, Hw) \). From this we obtain
\[
\text{Re}(Hv, w) = \text{Re}(iv, Hw).
\]
This implies $\text{Im}(Hv, w) = \text{Im}(v, Hw)$. Putting these together gives $\langle Hv, w \rangle = \langle v, Hw \rangle$. Hence $H$ is complex self-adjoint and $\langle Hz, z \rangle > 0$ for $z \neq 0$. □

**Lemma 2.2.** Let $w \in V$. Then $\langle Aw, w \rangle = \langle Aw, w \rangle + \text{Im}(Kw, w)$ and $\langle Aw, w \rangle = (Hw, w) + (Kw, w)$.

**Proof.** Let $w \in V$. Then

$$\langle Aw, w \rangle = \langle Hw, w \rangle + \langle Kw, w \rangle = \langle Hw, w \rangle + \text{Im}(Kw, w)$$

This implies the first statement. Taking the real part in the second line gives the second claim, which also follows directly from bilinearity of $(\cdot, \cdot)$.

Denote by $\det_V$ the determinant of a $\mathbb{R}$-linear map on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Let $d\mu_A(z) = \pi^{-n} \sqrt{\det_V A} e^{-(A_z, z)} dxdy$ and let $F_A$ be the space of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$||F||_A^2 := \int |F(z)|^2 d\mu_A < \infty.$$  

Our normalization of $d\mu$ is chosen so that $||1||_A = 1$. Just as in the classical case one can show that $F_A$ is a reproducing Hilbert space, but this will also follow from the following Lemma. We notice that all the holomorphic polynomials $p(z)$ are in $F$. To simplify the notation, we let $T_1 = H^{-1/2}$. Then $T_1$ is symmetric, positive definite and complex linear. Let $c_A = \sqrt{\det_V (A^{1/2} T_1)} = (\det_V (A)/\det_V (H))^{1/4}$.

**Lemma 2.3.** Let $F : V \to \mathbb{C}$ be holomorphic. Then $F \in F_A$ if and only if $F \circ T_1 \in F$ and the map $\Psi : F \to F_A$ given by

$$\Psi(F)(w) := c_A \exp \left(-\frac{(KT_1 w, T_1 w)}{2}\right) F(T_1 w)$$

is a unitary isomorphism. In particular

$$\Psi^* F(w) = \Psi^{-1} F(w) = c_A^{-1} \exp \left(\frac{(Kw, w)}{2}\right) F(\sqrt{F} w).$$

**Proof.** Let $F : V \to \mathbb{C}$. Then $F$ is holomorphic if and only if $F \circ T_1$ is holomorphic as $T_1$ is complex linear and invertible. Moreover, we also have:

$$||\Psi F||^2 = \pi^{-n} \int |\Psi F(w)|^2 e^{-(w, w)} dw$$

$$= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-(Kw, w)} e^{-((\sqrt{F} w, \sqrt{F} w))} dw$$

$$= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-(Kw, w)} e^{-((H+K)w, w)} dw$$

$$= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-(Aw, w)} dw$$

$$= ||F||_A^2$$

and thus, by polarization, $\Psi$ is unitary. □
Theorem 2.4. The space \( \mathbf{F}_A \) is a reproducing Hilbert space with reproducing kernel

\[
K_A(z, w) = c_A^{-2} e^{i(K(z,z) - (Hz, w))} e^{rac{1}{2} (K(w, w))}.
\]

Proof. By Lemma 2.3 we get

\[
c_A \exp(-\langle K \hat{T}_1 w, \hat{T}_1 w \rangle / 2) F(T_1 w) = \Psi(F)(w) = (\Psi(F), K)_{\mathbf{F}_A} = (K, \Psi^*(w))_F.
\]

Hence

\[
K_A(z, w) = c_A^{-1} \exp(\langle K w, w \rangle / 2) \Psi^*(K \sqrt{\mathbb{V}}_w) = c_A^{-2} e^{i(K(z,z) - (Hz, w))} e^{rac{1}{2} (K(w, w))}.
\]

\[\square\]

3. The Restriction Map

We assume as before that \( A > 0 \). We notice that Lemma 2.3 gives a unitary isomorphism \( \Psi U : L^2(\mathbb{R}^n) \to \mathbf{F}_A \), where \( U \) is the classical Segal-Bargmann transform. But this is not the natural transform that we are looking for. As \( H \) is positive definite there is an orthonormal basis \( e_1, \ldots, e_n \) of \( V \) and positive numbers \( \lambda_j > 0 \) such that \( He_j = \lambda_j e_j \). Let \( V_\mathbb{R} := \sum \mathbb{R} e_k \). Set \( \sigma(\sum a_i e_i) = \sum \bar{a}_i e_i \). Then \( \sigma \) is a conjugation with \( V_\mathbb{R} = \{ z : \sigma z = \bar{z} \} \). We say that a vector is real if it belongs to \( V_\mathbb{R} \). As \( He_j = \lambda_j e_j \) with \( \lambda_j \in \mathbb{R} \) it follows that \( HV_\mathbb{R} \subseteq V_\mathbb{R} \). We denote by det the determinant of a \( \mathbb{R} \)-linear map of \( V_\mathbb{R} \).

Lemma 3.1. \( \langle K z, w \rangle = \langle K w, z \rangle \).

Proof. Note that \( \sigma K \) is complex linear. Since \( J^* = -J, K = \frac{1}{2}(A - JA_J^{-1}) \) is real symmetric, Thus \( \langle K w, z \rangle = \langle w, K z \rangle = \langle K z, w \rangle \). Also note \( \langle i K z, w \rangle = \langle J K z, w \rangle = \langle K J z, w \rangle = \langle J z, K w \rangle = -\langle i z, K w \rangle \). Hence \( \text{Re}(i K z, w) = -\text{Re}(i z, K w) \). So \( -\text{Im}(K z, w) = \text{Im}(z, K w) \). This gives \( \text{Im}(K w, z) = \text{Im}(K z, w) \). Hence \( \langle K z, w \rangle = \langle K w, z \rangle \). \[\square\]

Lemma 3.2. \( (\sigma K)^* = K \sigma \).

Proof. We have \( \langle \sigma z, \sigma w \rangle = \langle w, z \rangle \). Hence

\[
\langle \sigma K z, w \rangle = \langle \sigma w, \sigma^2 K z \rangle = \langle \sigma w, K z \rangle = \langle z, K \sigma w \rangle.
\]

\[\square\]

Corollary 3.3. If \( x, y \in V_\mathbb{R} \), then \( \langle H x, y \rangle \) is real and \( \langle A x, y \rangle = \langle A y, x \rangle \).

Proof. Clearly \( \langle - \rangle \) is real on \( V_\mathbb{R} \times V_\mathbb{R} \). Since \( HV_\mathbb{R} \subseteq V_\mathbb{R} \), we see that \( \langle H x, y \rangle \) is real. Next, \( \langle A x, y \rangle = \langle H x, y \rangle + \langle K x, y \rangle \). The term \( \langle H x, y \rangle \) equals \( \langle H y, x \rangle \) because \( \langle H x, y \rangle \) is real and \( H \) is self-adjoint. On the other hand, \( \langle K x, y \rangle = \langle K y, x \rangle \) by Lemma 3.1. So \( \langle A x, y \rangle = \langle A y, x \rangle \). \[\square\]

Lemma 3.4. Define \( m : V_\mathbb{R} \times V \to \mathbb{C} \) by \( m(x, z) = e^{i(H x, x)} e^{i(K z, z)} e^{-(A x, x)/2} \). Then \( m \) is a multiplier. Moreover, if \( T_x F(z) := m(x, z) F(z - x) \), then \( x \mapsto T_x \) is a representation of the abelian group \( V_\mathbb{R} \) on \( \mathbf{F}_A \). It is unitary if \( KV_\mathbb{R} \subseteq V_\mathbb{R} \).
Proof. We first show $m$ is a multiplier:

$$m(x, z) m(y, z - x) = e^{i(\mathcal{H}z, x)} e^{\langle A x, x \rangle / 2} e^{i(\mathcal{H}(z - x), y)} e^{\langle K(z - x), y \rangle} e^{-\langle A y, y \rangle / 2}$$

$$= e^{i(\mathcal{H}z, x + y)} e^{\langle K z, x + y \rangle} e^{-\langle A y, y \rangle} e^{\langle A z, x \rangle / 2} e^{-\langle A y, y \rangle / 2}$$

$$= e^{i(\mathcal{H}z, x + y)} e^{\langle K z, x + y \rangle} e^{-\langle A(x + y), x + y \rangle / 2}$$

$$= m(x + y, z).$$

Since $m$ is a multiplier, we have $T_x T_y = T_{x+y}$. For each $T_x$ to be unitary, we need $|m(x, z)| = e^{\langle A z, x \rangle - \langle A x, x \rangle / 2}$. But

$$|m(x, z)| = e^{i(\mathcal{H}z, x)} e^{\langle K z, x \rangle} e^{-\langle A x, x \rangle / 2} = e^{\langle A z, x \rangle - \langle A x, x \rangle / 2} e^{i(\mathcal{H}z, x - K z, x)}.$$  

Thus $T_x$ is unitary for all $x$ if and only if the real part of every vector $K \mathcal{Z} - K z$ is 0. Since $\mathcal{Z} - z$ runs over $iV_\mathcal{R}$ as $z$ runs over $V$, $T_x$ is unitary for all $x$ if and only if $K'(iV_\mathcal{R}) \subset iV_\mathcal{R}$, which is equivalent to $K(V_\mathcal{R}) \subset V_\mathcal{R}$.

Notice that $\det V H = (\det H)^2$. To simplify some calculations later on we define $c := (2\pi)^{-n/4} (\frac{\det V A}{\det H})^{1/4}$. We remark for further reference:

Lemma 3.5. $c^{-2} \mathcal{C}^2 = \sqrt{\frac{\det H}{(2\pi)^{n/2}}} \text{ and } c^{-1} \frac{\sqrt{\det H}}{(2\pi)^{n/4}} = \left(\frac{2}{\pi}\right)^{n/4} \frac{(\det H)^{3/4}}{(\det V A)^{1/4}}$.

Let $D(x) = cm(x, 0) = c e^{\langle A x, x \rangle / 2}$ and define $R : F_A \rightarrow C^\infty(V_\mathcal{R})$ by $RF(x) := D(x)F(x)$. Extending the bilinear form $x \mapsto \langle A x, x \rangle$ to a complex bilinear form $\langle x, z \rangle_A$ on $V$ shows that $D$ has a holomorphic extension to $V$.

Lemma 3.6. The restriction map $R$ intertwines the action of $V_\mathcal{R}$ on $F_A$ and the left regular action $L$ on functions on $V_\mathcal{R}$.

Proof. We have

$$R(T_y F)(x) = cm(x, 0) T_y F(x)$$

$$= cm(x, 0) m(y, x) F(x - y)$$

$$= cm(x, 0) m(-y, -x) F(x - y)$$

$$= cm(x - y, 0) F(x - y)$$

$$= Ly RF(x).$$

\[ \square \]

4. The Generalized Segal–Bargmann Transform

As for the classical space, $R$ is a densely defined, closed operator. It also has dense image in $L^2(V_\mathcal{R})$. To see this, let $\{h_\alpha\}_\alpha$ be the orthonormal basis of $L^2(V_\mathcal{R})$ given by the Hermite functions. Then $\{\det(A)^{1/2} h_\alpha(\sqrt{A} x)\}_\alpha$ is an orthonormal basis of $L^2(V_\mathcal{R})$ which is contained in the image of $R$. It follows again that $R$ has an adjoint and

$$R^* h(z) = (R^* h, K_{A, z}) = (h, R K_{A, z})$$
where $K_{A,z}(w) = K_A(w, z) = c_A^2 e^{\frac{i}{2} (K(w, w) - (H w, z))} e^{\frac{i}{2} (K z, z)}$. Thus

$$R^* h(z) = c \int h(x) e^{-(Ax,x)/2} K_A(x, z) dx$$

$$= c_A^{-2} e \int h(x) e^{-(Ax,x)/2} e^{\frac{i}{2} (K z, x)} e^{\frac{i}{2} (K x, z)} dx$$

$$= c_A^{-2} e e^{\frac{i}{2} (K z, z)} \int h(x) e^{-(Ax,x)/2} e^{-(H x, x)/2} e^{\frac{i}{2} (K x, z)} dx$$

$$= c_A^{-1} e^{\frac{i}{2} (K z, z)} \int h(x) e^{-(Ax,x)/2} e^{(z, H x)/2} dx$$

$$= c_A^{-2} e^{\frac{i}{2} (K z, z)} e^{\frac{i}{2} (z, H x)} \int h(x) e^{-(Ax,x)/2} e^{-(H x, H x)/2} dx$$

for $(z, H x) = (H x, z) = (x, H z)$ and $(x, H x) = (z, H z)$. Thus we finally arrive at

$$R^* h(z) = c_A^{-2} e^{\frac{i}{2} (z, H x + K z)} e^{\frac{i}{2} (z, H x)} h(z).$$

Let $P : V_{\mathbb{R}} \to V_{\mathbb{R}}$ be positive. Define $\phi_P(x) = \sqrt{\det(P)}/(2\pi)^{-n/2} e^{-\||P x||^2/2}$. For $t > 0$, let $P(t) = P/t$.

**Lemma 4.1.** Let the notation be as above. Then $0 < t \mapsto \phi_P(t)$ is a convolution semigroup, i.e., $\phi_P(t+s) = \phi_P(t) * \phi_P(s)$.

**Proof.** This follows by change of parameters $y = \sqrt{P} x$ from the fact that $\phi_{(I+t)P}(x) = (2\pi t)^{-n/2} e^{-\||P x||^2/2t}$ is a convolution semigroup. \qed

We define a unitary operator $W$ on $L^2(V_{\mathbb{R}})$ by

$$W f(x) = e^{i \text{Im}(x, K x)} f(x) = e^{i \text{Im}(x, A x)} f(x).$$

We know $W = I$ if $K V_{\mathbb{R}} \subseteq V_{\mathbb{R}}$ and this occurs if $A$ leaves $V_{\mathbb{R}}$ invariant.

**Lemma 4.2.** Let $h$ be in the domain of definition of $R^*$. Then $RR^* h = W(\phi_H * h)$.

**Proof.** We notice first that $c_A^{-2} e^{2} = (2\pi)^{-n/2} \sqrt{H}$ by Lemma 3.5. From (4.1) we then get

$$RR^* h(x) = c e^{-\frac{i}{2} (Ax,x)} R^* h(x)$$

$$= c_A^{-2} e^{-\frac{i}{2} (Ax,x)} e^{\frac{i}{2} (x, H x + K x)} e^{-\frac{i}{2} (y, H y)} * h(x)$$

$$= (2\pi)^{-n/2} \sqrt{\det(H)} e^{-\frac{i}{2} (Ax,x)} e^{\frac{i}{2} (x, A x)} e^{-\frac{i}{2} (y, H y)} * h(x)$$

$$= (2\pi)^{-n/2} \sqrt{\det(H)} e^{i \text{Im}(x, A x)} \int e^{-\frac{i}{2} (y, H y)} h(x - y) dy$$

$$= (2\pi)^{-n/2} \sqrt{\det(H)} e^{i \text{Im}(x, A x)} \int e^{-\frac{i}{2} ||x||^2} h(x - y) y$$

$$= W(\phi_H * h)(x)$$

\qed

Lemma 4.1 and Lemma 4.2 leads to the following corollary:
Corollary 4.3. Suppose $AV_\mathbb{R} \subseteq V_\mathbb{R}$. Then

$$|R^*| h(x) = \phi_{H(1/2)} * h(x) = \frac{\sqrt{\det(H)}}{\pi^{n/2}} \int_{V_\mathbb{R}} e^{-||\sqrt{\Pi y}||^2} h(x - y) \, dy.$$ 

Theorem 4.4 (The Segal–Bargmann Transform). Suppose $A$ leaves $V_\mathbb{R}$ invariant. Then the operator $U_A : L^2(V_\mathbb{R}) \to F_A$ defined by

$$U_A f(z) = \left( \frac{2}{\pi} \right)^{n/4} \frac{\det H)^{3/4}}{\det A} \frac{1}{4} e^{\frac{1}{2}(\langle Hz, \sigma \rangle + \langle z, Kz \rangle)} \int e^{i(H(z - y), \sigma - y)} f(y) \, dy.$$ 

is a unitary isomorphism. The map $U_A$ is called the generalized Segal–Bargmann transform.

Proof. By polarization we can write $R^* = U |R^*|$ where $U : L^2(V_\mathbb{R}) \to F_A$ is a unitary isomorphism. Taking adjoints gives $|R^*| U^* = R$. Hence $RU = |R^*|$. Hence

$$cm(x) Uh(x) = RU h(x)$$

$$= |R^*| h(x)$$

$$= \frac{\sqrt{\det(H)}}{\pi^{n/2}} \int_{V_\mathbb{R}} e^{-||\sqrt{\Pi y}||^2} h(x - y) \, dy.$$ 

Since $m(x) = e^{-\frac{1}{2}(\langle x, Hx \rangle + \langle x, Kx \rangle)}$, we have using Lemma 3.5:

$$U f(z) = \left( \frac{2}{\pi} \right)^{n/4} \frac{\det H)^{3/4}}{\det A} \frac{1}{4} e^{\frac{1}{2}(\langle Hz, \sigma \rangle + \langle z, Kz \rangle)} \int e^{i(H(z - y), \sigma - y)} f(y) \, dy.$$ 

By holomorphicity, this implies

$$U f(z) = \left( \frac{2}{\pi} \right)^{n/4} \frac{\det H)^{3/4}}{\det A} \frac{1}{4} e^{\frac{1}{2}(\langle Hz, \sigma \rangle + \langle z, Kz \rangle)} \int e^{i(H(z - y), \sigma - y)} f(y) \, dy$$

is the Bargmann–Segal transform. \hfill \Box

5. The Gaussian Formulation

In infinite dimensions, there is no useful notion of Lebesgue measure but Gaussian measure does make sense. So, with a view to extension to infinite dimensions, we will recast our generalized Segal–Bargmann transform using Gaussian measure instead of Lebesgue measure as the background measure on $V_\mathbb{R}$. Of course, we have already defined the Fock space $F_A$ using Gaussian measure.

As before, $V$ is a finite-dimensional complex vector space with Hermitian inner-product $\langle \cdot, \cdot \rangle$, and $A : V \to V$ is a real-linear map which is symmetric, positive-definite with respect to the real inner-product $\langle \cdot, \cdot \rangle = \text{Re}(\langle \cdot, \cdot \rangle)$, i.e. $(Az, z) > 0$ for all $z \in V$ except $z = 0$. We assume, furthermore, that there is a real subspace $V_\mathbb{R}$ for which $V = V_\mathbb{R} + iV_\mathbb{R}$, the inner-product $\langle \cdot, \cdot \rangle$ is real-valued on $V_\mathbb{R}$ and $A(V_\mathbb{R}) \subset V_\mathbb{R}$. As usual, $A$ is the sum

$$A = H + K$$

where $H = (A - iAi)/2$ is complex-linear on $V$ and $K = (A + iAi)/2$ is complex-conjugate-linear. The real subspaces $V_\mathbb{R}$ and $iV_\mathbb{R}$ are $\langle \cdot, \cdot \rangle$-orthogonal because for any $x, y \in V_\mathbb{R}$ we have $(x, iy) = \text{Re}(x, iy) = -\text{Re}(i(x, y))$, since $(x, y)$ is real, by...
hypothesis. Since $A$ preserves $V_{\mathbb{R}}$ and is symmetric, it also preserves the orthogonal complement $iV_{\mathbb{R}}$. Thus $A$ has the block diagonal form
\begin{equation}
A = \begin{bmatrix} R & 0 \\ 0 & T \end{bmatrix} = d(X, Y)
\end{equation}

Here, and henceforth, we use the notation $d(X, Y)$ to mean the real-linear map $V \to V$ given by $a \mapsto Xa$ and $ia \mapsto iYa$ for all $a \in V_{\mathbb{R}}$, where $X, Y$ are real-linear operators on $V_{\mathbb{R}}$. Note that $d(X, Y)$ is complex-linear if and only if $Y = -X$. The operator $d(X, Y)$ is the unique complex-linear map $V \to V$ which restricts to $X$ on $V_{\mathbb{R}}$, and we will denote it by $X_V$:
\begin{equation}
X_V = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}
\end{equation}

The hypothesis that $A$ is symmetric and positive-definite (by which we mean $A > 0$, not just $A \geq 0$) means that $R$ and $T$ are symmetric, positive definite on $V_{\mathbb{R}}$. Consequently, the real-linear operator $S$ on $V_{\mathbb{R}}$ given by
\begin{equation}
S = 2(R^{-1} + T^{-1})^{-1}
\end{equation}
is also symmetric, positive-definite.

The operators $H$ and $K$ on $V$ are given by
\begin{equation}
H = \frac{1}{2}(R_V + T_V), \quad K = d \left( \frac{1}{2}(R - T), \frac{1}{2}(T - R) \right)
\end{equation}
Using the conjugation map
\[ \sigma : V \to V : a + ib \mapsto a - ib \quad \text{for } a, b \in V_{\mathbb{R}} \]
we can also write $K$ as
\begin{equation}
K = \frac{1}{2}(R_V - T_V)\sigma
\end{equation}
Now consider the holomorphic functions $\rho_T$ and $\rho_S$ on $V$ given by
\begin{equation}
\rho_T(z) = \frac{(\det T)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(T_{Vz}, z)}
\end{equation}
\begin{equation}
\rho_S(z) = \frac{(\det S)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(S_{Vz}, z)}
\end{equation}
where $n = \dim V_{\mathbb{R}}$. Restricted to $V_{\mathbb{R}}$, these are density functions for Gaussian probability measures.

The Segal-Bargmann transform in this setting is given by the map
\begin{equation}
S_A : L^2(V_{\mathbb{R}}, \rho_S(x)dx) \to F_A : f \mapsto S_A f
\end{equation}
where
\begin{equation}
S_A f(z) = \int_{V_{\mathbb{R}}} f(x) \rho_T(z - x) \, dx = \int_{V_{\mathbb{R}}} f(x) c(x, z) \rho_S(x) \, dx
\end{equation}
where the generalized “coherent state” function $c$ is specified, for $x \in V_{\mathbb{R}}$ and $z \in V$, by
\begin{equation}
c(x, z) = \frac{\rho_T(x - z)}{\rho_S(x)}
\end{equation}
It is possible to take (5.9) as the starting point, with $f \in L^2(V_\mathbb{R}, \rho_S(x)dx)$ and prove that: (i) $S_A f(z)$ is well-defined, (ii) $S_A f$ is in $F_A$, (iii) $S_A$ is a unitary isomorphism onto $F_A$. However, we shall not work out everything in this approach since we have essentially proven all this in the preceding sections. Full details of a direct approach would be obtained by generalizing the procedure used in [12]. In the present discussion we shall work out only some of the properties of $S_A$.

**Lemma 5.1.** Let $w, z \in V$. Then:

(i) The function $x \mapsto c(x, z)$ belongs to $L^2(V_\mathbb{R}, \rho_S(x)dx)$, thereby ensuring that the integral (5.9) defining $S_A f(z)$ is well-defined;

(ii) The $S_A$-transform of $c(\cdot, w)$ is $K_A(\cdot, \overline{w})$:

\begin{equation}
[S_A c(\cdot, w)](z) = K_A(z, \overline{w})
\end{equation}

and so, in particular,

\begin{equation}
K_A(z, w) = \int_{V_\mathbb{R}} \frac{\rho_T(x-z)\rho_T(x-w)}{\rho_S(x)} \, dx
\end{equation}

(iii) The transform $S_A$ preserves inner-products on the linear span of the functions $c(\cdot, w)$:

$$
\langle c(\gamma, w), c(\gamma, z) \rangle_{L^2(V_\mathbb{R}, \rho_S(x)dx)} = K_A(w, z) = \langle K_A(\gamma, \overline{w}), K_A(\gamma, \overline{z}) \rangle_{F_A}
$$

**Proof.** (i) is equivalent to finiteness of $\int_{V_\mathbb{R}} \frac{|c(x, z)|^2}{\rho_S(x)} \, dx$, which is equivalent to positivity of the operator $2T - S$. To see that $2T - S$ is positive observe that

\begin{align*}
2T - S &= 2T[(R^{-1} + T^{-1}) - T^{-1}](R^{-1} + T^{-1})^{-1} \\
&= 2T R^{-1} (R^{-1} + T^{-1})^{-1} = T R^{-1} S \\
&= 2(T^{-1} + T^{-1} R T^{-1})^{-1}
\end{align*}

and in this last line $T^{-1} > 0$ (being the inverse of $T > 0$) and $(T^{-1} R T^{-1} x, x) = (R T^{-1} x, T^{-1} x) \geq 0$ by positivity of $R$. Thus $2T - S$ is positive, being twice the inverse of the positive operator $T^{-1} + T^{-1} R T^{-1}$.

(ii) is the result of a lengthy calculation which, despite an unpromising start, leads from complicated expressions to simple ones. To avoid writing a lot of complex conjugates we shall use the symmetric complex bilinear pairing $v \cdot w = \langle v, \overline{w} \rangle$ for $v, w \in V$, writing $v^2$ for $vv$. More seriously, we shall denote the complex-linear operator $T v$ which restricts to $T$ on $V_\mathbb{R}$ simply by $T$. It is readily checked that $T$ continues to be symmetric in the sense that $T v \cdot w = v \cdot T w$ for all $v, w \in V$. We start with

\begin{align*}
a &\overset{\text{def}}{=} [S_A c(\cdot, w)](z) \\
&= \int_{V_\mathbb{R}} \frac{\rho_T(x-w)\rho_T(z-x)}{\rho_S(x)} \, dx \\
&= (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V_\mathbb{R}} e^{-\frac{i}{4}[2T(x-w) \cdot (x-z) + T(x-z)(x-z) - S(x^2)]} \, dx \\
&= (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V_\mathbb{R}} e^{-\frac{i}{4}(2T-S)x \cdot x - 2T(x-w) \cdot (x+z) + T w \cdot w + T z \cdot z)} \, dx
\end{align*}
Recall from the proof of (i) that \(2T - S > 0\). For notational simplicity let \(L = (2T - S)^{1/2}\) and \(M = L^{-1}T\). Then

\[
a = (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}(\det L)} \int_{V_0} e^{-\frac{1}{2}(Mx - M(w + z))^2} dx e^{-\frac{1}{4}[Tw \cdot w + Tz \cdot z - M(w + z) \cdot M(w + z)]} e^{-\frac{1}{4}[Tw + Tz - M(w + z) \cdot M(w + z)]}
\]

To simplify the last exponent observe that

\[
Tw \cdot w - Mw \cdot Mw = Tw \cdot w - T(w \cdot L^{-2}T)w
\]

\[
= Tw \cdot w - T \cdot (2T - S)^{-1}Tw
\]

\[
= Tw \cdot w - \frac{1}{2}Tw \cdot (T^{-1} + T^{-1}RT^{-1})Tw \quad \text{using (5.13)}
\]

\[
= Tw \cdot w - \frac{1}{2}Tw \cdot (w + T^{-1}Rw)
\]

\[
= \frac{1}{2}(Tw \cdot w - Rw \cdot w)
\]

\[
= -\langle K\vartheta, \vartheta \rangle \quad \text{by (5.5)}
\]

The same holds with \(z\) in place of \(w\). For the “cross term” we have

\[
Mw \cdot Mz = Tw \cdot L^{-2}Tz
\]

\[
= Tw \cdot (2T - S)^{-1}Tz
\]

\[
= \frac{1}{2}Tw \cdot (T^{-1} + T^{-1}RT^{-1})Tz
\]

\[
= \frac{1}{2}(Tw \cdot z + w \cdot Rz)
\]

\[
= 2w \cdot Hz
\]

Putting everything together we have

\[
[S_{AC}(\cdot, w)](z) = \frac{\det T}{(\det S)^{1/2}(\det L)} e^{-\frac{1}{2}(Kw, w)} e^{-\frac{1}{2}(Hz, w)} e^{-\frac{1}{2}(Kz, z)}
\]

In Lemma 6.2 below we prove that

\[
\frac{\det T}{(\det S)^{1/2}(\det L)} = \left(\frac{\det V(A)}{\det V(H)}\right)^{-1/2} = c_A^{-2}
\]

So

\[
[S_{AC}(\cdot, w)](z) = K_A(w, z)
\]

For (iii), we have first:

\[
\langle c(\cdot, w), c(\cdot, z) \rangle_{L^2(\rho_d(\cdot)dx)} = [S_{AC}(\cdot, w)](z) = K_A(z, \vartheta) = K_A(w, z)
\]

The second equality in (iii) follows from the fact that \(K_A\) is a reproducing kernel.

\[
\square
\]

6. The Evaluation Map and Determinant Relations

Recall the reproducing kernel

\[
K_A(z, w) = c_A^{-2} e^{-\frac{1}{2}(z, Kz) + \frac{1}{2}(Kw, w) + (Hz, w)}
\]
where
\[ c_A^{-2} = \left( \frac{\det_V H}{\det_V A} \right)^2 \]

Being a reproducing kernel for \( F_A \) means
\[
f(w) = (f, K_A(\cdot, w)) = \pi^{-n}(\det A)^{1/2} \int_V f(z) K_A(w, z) \, |dz|
\]
where \( |dz| = dx\,dy \) signifies integration with respect to Lebesgue measure on the real inner-product space \( V \). Thus we have

**Proposition 6.1.** For any \( z \in V \), evaluation map
\[
\delta_z : F_A \to \mathbb{C} : f \mapsto f(z)
\]
is bounded linear functional with norm
\[
\|\delta_z\| = K_A(z, z)^{1/2} = c_A^{-1} e^{(A_z, z)}
\]

**Proof.** We have
\[
|\delta_z f| = |f(z)| = |(f, K_A(\cdot, z))| \leq \|f\|_{F_A} K_A(z, z)^{1/2}
\]
because, again by the reproducing kernel property we have
\[
\|K_A(\cdot, z)\|_{F_A}^2 = (K_A(\cdot, z), K_A(\cdot, z))_{F_A} = K_A(z, z)
\]
This last calculation also shows that the inequality in (6.3) is an equality of \( f = K_A(\cdot, z) \) and thereby shows that \( \|\delta_z\| \) is actually equal to \( K_A(z, z)^{1/2} \). The latter is readily checked to be equal to \( c_A^{-1} e^{(A_z, z)} \). \( \square \)

Next we make two observations about the constant \( c_A \), the first of which has already been used.

**Lemma 6.2.** For the constant \( c_A \) we have
\[
c_A^{-2} = \left( \frac{\det_V H}{\det_V A} \right)^2 = \frac{\det T}{(\det S)^{1/2} \det L}
\]
where, as before, \( L = (2T - S)^{1/2} \) and \( S = 2(R^{-1} + T^{-1})^{-1} \).

**Proof.** Recall from (5.13) that \( 2T - S = TR^{-1}S \). Note also that
\[
S^{-1} = \frac{1}{2}(R^{-1} + T^{-1}) = R^{-1} \frac{R + T}{2} T^{-1} = R^{-1}(H_{V_R} T^{-1}
\]
So
\[
\left( \frac{\det_V A}{\det_V H} \right)^{1/2} \frac{\det T}{(\det S)^{1/2} \det L} = \frac{(\det R)^{1/2} (\det T)^{1/2}}{(\det S)^{1/2} \det T} \frac{\det T}{\det S^{-1} \det R \det T (\det S)^{1/2} \det T^{1/2} \det R^{-1/2} \det S^{1/2}} = 1
\]
which implies the desired result. \( \square \)

Next we prove a determinant relation which implies \( c_A \geq 1 \):

**Lemma 6.3.** If \( R \) and \( T \) are positive definite \( n \times n \) matrices (symmetric if real) then
\[
\sqrt{\det R} \det T \leq \det \left( \frac{R + T}{2} \right)
\]
with equality if and only if \( R = T \).
Proof. Note first that the matrix
\begin{equation}
D \overset{\text{def}}{=} R^{-1/2}TR^{-1/2}
\end{equation}
is positive definite because \((R^{-1/2}TR^{-1/2}x, x) = (TR^{-1/2}x, R^{-1/2}x) \geq 0\) since 
\(T > 0\), with equality if and only if \(R^{1/2}x = 0\) if and only if \(x = 0\). So \(D = (R^{-1/2}TR^{-1/2})^1/4\) makes sense and is also positive definite (and is symmetric if we 
are working with reals). We have then
\[
\frac{\det R \det T}{(\det \frac{R+T}{2})^2} = \frac{\det R \det (R^{1/2}D^4R^{1/2})}{[\det R^{1/2}(\frac{1}{2}D^2) R^{1/2}]^2} \\
= \left[ \det \left( \frac{D^2 + D^{-2}}{2} \right) \right]^{-2} \\
= \left[ \det \left( I + \left( \frac{1}{\sqrt{2}}D - \frac{1}{\sqrt{2}}D^{-1} \right)^2 \right) \right]^{-2}
\]
To summarize:
\begin{equation}
\frac{\det R \det T}{(\det \frac{R+T}{2})^2} = \left[ \det \left( I + \left( \frac{1}{\sqrt{2}}D - \frac{1}{\sqrt{2}}D^{-1} \right)^2 \right) \right]^{-2}
\end{equation}
where \(D = (R^{-1/2}TR^{-1/2})^1/4\). Diagonalizing \(D\) makes it apparent that this last 
term is \(< 1\) with equality if and only if \(D = D^{-1}\), which is equivalent to \(D^4 = I\) 
which holds if and only if \(R = T\). \(\square\)

As consequence we have for \(c_A\):
\[
c_A = \left( \frac{\det V A}{\det V H} \right)^{1/4} = \left( \frac{\det R \det T}{(\det \frac{R+T}{2})^2} \right)^{1/4} = \left( \frac{\sqrt{\det R \det T}}{\det \frac{R+T}{2}} \right)^{1/2}
\]
and so
\begin{equation}
c_A^{-2} = \frac{\det \frac{R+T}{2}}{\sqrt{\det R \det T}} \geq 1
\end{equation}
with equality if and only if \(R = T\).

When extending this theory to infinite-dimensional we have to note that in order 
to retain a meaningful notion of evaluation \(\delta_z : f \rightarrow f(z)\), the constant \(c_A^{-1}\) which 
appears in the norm \(||\delta_z||\) given in (6.2) must be finite. The expression for \(c_A^{-2}\) 
 obtained from (6.6) gives a more explicit condition on \(R\) and \(T\) for this finiteness 
to hold.

If \(R\) and \(T\) are both scalar operators, say \(R = rf\) and \(T = rT\), then (6.7) shows 
that \(c_A^{-1}\) equals \(|(r + t)/(2\sqrt{r}t)|^{n/2}\) which is bounded as \(n \rightarrow \infty\) if and only if \(r = t\). 
This observation was made in [12].

7. REMARKS ON EXTENSION TO INFINITE DIMENSIONS

The Gaussian formulation permits extension to the infinite-dimensional situation, 
at least with some conditions placed on \(A\) so as to make such an extension 
reasonable. Suppose then that \(V\) is an infinite-dimensional separable complex 
Hilbert space, \(V_R\) a real subspace on which the inner-product is real-valued, and 
\(A : V \rightarrow V\) a bounded symmetric, positive-definite real-linear operator carrying \(V_R\) 
into itself. The operators \(R, T, S, H\) and \(K\) are defined as before. Assume that \(R\)
and $T$ commute and that there is an orthonormal basis $e_1, e_2, \ldots$ of $V_\mathbb{R}$ consisting of simultaneous eigenvectors of $R$ and $T$ (greater generality may be possible but we discuss only this case). Let $V_n$ be the complex linear span of $e_1, \ldots, e_n$, and $V_{n,\mathbb{R}}$ the real linear span of $e_1, \ldots, e_n$. Then $A$ restricts to an operator $A_n$ on $V_n$, and we have similarly restrictions $H_n, K_n$ on $V_n$ and $R_n, T_n, S_n$ on $V_{n,\mathbb{R}}$. The unitary transform $S_A$ may be obtained as a limit of the finite-dimensional transforms $S_{A_n}$.

The Gaussian kernels $\rho_S$ and $\rho_T$ do not make sense anymore, and nor does the coherent state $c$, but the Gaussian measures $d\gamma_S(x) = \rho_S(x)dx$ and $\mu_A$ do have meaningful analogs. There is a probability space $V'_{\mathbb{R}}$, with a $\sigma$-algebra $\mathcal{F}$ on which there is a measure $\gamma_S$, and there is a linear map $V_{\mathbb{R}} \to L^2(V'_{\mathbb{R}}, \gamma_A) : x \mapsto G(x) = (x, \cdot)$, such that the $\sigma$-algebra $\mathcal{F}$ is generated by the random variables $G(x)$, and each $G(x)$ is (real) Gaussian with mean 0 and variance $S^{-1}x, x$). Similarly, there is probability space $V', \overline{V'}$, with a $\sigma$-algebra $\mathcal{F}_1$ on which there is a measure $\mu_A$, and there is a real-linear map $V \to L^2(V', \mu_A) : z \mapsto G_1(z) = (z, \cdot)$. Then for each $z \in V$, written as $z = a + ib$ with $a, b \in V_{\mathbb{R}}$, we have the complex-valued random variable on $V'$ given by

$$\hat{z} = G_1(a) + iG_1(b)$$

Suppose $g$ is a holomorphic function of $n$ complex variables such that

$$\int_V |g(\bar{e}_1, \ldots, \bar{e}_n)|^2 d\mu_A < \infty.$$ 

Define $F_A$ to be the closed linear span of all functions of the type $g(\bar{e}_1, \ldots, \bar{e}_n)$ in $L^2(\mu_A)$ for all $n \geq 1$. We may then define $S_A$ of a function $f \in L^2(\mu_A)$ to be $(S_A f)(\bar{e}_1, \ldots, \bar{e}_n)$, and then extend $S_A$ be continuity to all of $L^2(\gamma_S)$. In writing $(S_A f)$ we have identified $V_n$ with $\mathbb{C}^n$ and $V_{n, \mathbb{R}}$ with $\mathbb{R}^n$ using the basis $e_1, \ldots, e_n$.

A potentially significant application of the infinite-dimensional case would be to situations where $V_{\mathbb{R}}$ is a path space and $A$ is arises from a suitable differential operator. For the “classical case” where $R = T = t\mathcal{H}$ for some $t > 0$, this leads to the Hall transform [6] for Lie groups as well as the path-space version on Lie groups considered in [7].

References


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