THE SYMPLETIC SEMIGROUP AND RICCATI DIFFERENTIAL EQUATIONS: A CASE STUDY

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ABSTRACT. In this paper we study close connections that exist between the Riccati operator (differential) equation that arises in linear control systems and the sympletic group and its subsemigroup of sympletic Hamiltonian operators. A canonical triple factorization is derived for the sympletic Hamiltonian operators, and their closure under multiplication is deduced from this property. This semigroup of Hamiltonian operators, which we call the sympletic semigroup, is studied from the viewpoint of Lie semigroup theory, and resulting consequences for the theory of the Riccati equation are delineated. Among other things, these developments provide an elementary proof for the existence of a solution for the Riccati equation for all $t \geq 0$ under rather general hypotheses.

1. INTRODUCTION

The main purpose of this paper is to demonstrate how the Lie theory of subsemigroups of a matrix group, or more generally a Lie group, can be applied to problems in geometric control theory. We have chosen to do this in the form of a case study of a basic and familiar setting in control theory: the familiar Riccati equation that arises in the context of linear control systems with quadratic costs. The bulk of our theory carries through in the infinite dimensional setting with little additional effort, and we develop our theory in this context. This generalization is perhaps of some interest since both classical control theory and the Lie theory of semigroups have typically been developed in the finite dimensional setting.

We recall the primary connection of Lie semigroup theory with geometric control theory. Suppose that the states of a control problem are points of a Lie group and the controls are right invariant vector fields, or that the control problem can be reinterpreted so that this is the case. If the control functions are closed under concatenation,

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then the attainable set from the identity forms an infinitesimally generated subsemigroup of the Lie group, and all attainable sets are translates of this semigroup. If this attainability semigroup is closed in the group, then it is an example of a Lie semigroup (i.e., a closed infinitesimally generated semigroup). In addition to the techniques of geometric control, one has available the vast machinery and structure theory of Lie groups and Lie algebras to study control problems on Lie groups and the attainability semigroup. The attainability semigroup has been used primarily to study questions of controllability (see, for example, the survey [11]), but in this paper we seek to make a case that a detailed understanding of the attainability semigroup can be useful for attacking a variety of control questions. We refer the reader to the references [4] and [5] for the theory of Lie semigroups; a good background article on semigroups and control is the survey article [8].

Connections between linear control theory, the Riccati equation, and the sympletic group are well known; see, for example Hermann [3], Shayman [12], and Jurdjevic [6, Chapter 8] and the references cited in those sources. In this paper we focus on connections to the symplectic subsemigroup, which consists of those sympletic transformations that are sometimes called Hamiltonian. This semigroup has largely been overlooked in the control context; see, however, Bougerol [1], which was an important inspiration for our investigations. We exploit properties of the sympletic group and sympletic semigroup both to rederive some familiar results concerning the Riccati equation from this vantage point, hopefully with some new insights along the way, and to further extend and generalize the theory. We employ (and thus illustrate) a variety of basic tools from Lie group and Lie semigroup theory such as pushing forward control systems from groups to homogeneous spaces (Section 5) and the subtangential set of a semigroup, called its Lie wedge (Section 8). But the primary structure theorem for the symplectic semigroup, which is crucial to many of our applications, is its triple decomposition as given in Section 6. The triple decomposition often allows us to break up problems into much simpler subcases.

An important order exists on the symmetric operators called the Loewner order. In the last two sections we consider this order and its connections with the sympletic semigroup and the Riccati equation.

Many of the results of this paper are not new, but rather are new derivations of known results from the perspective of Lie group and Lie semigroup theory. Part of the purpose, as already mentioned, is to give an accessible case study of Lie semigroup theory and its connections with control theory. However, this paper is also foundational for more advanced and original applications of the sympletic semigroup to the study of Riccati equations that we plan to publish in a subsequent paper or papers.

2. Symplectic Spaces and the Sympletic Group

In this section we recall basic results concerning sympletic spaces and the sympletic group. These results are well-known, particularly in the finite dimensional setting, but it will be convenient to have them at hand for the general setting of this paper. Crucial for later purposes are the familiar results Proposition 2.5 through Proposition 2.7 at the end of the section, and the reader may choose simply to glance at them and move on.

Let V be a vector space over $\mathbb{F} = \mathbb{R}$, the real numbers, or $\mathbb{F} = \mathbb{C}$, the complex numbers. A *symplectic form* on V is a nondegenerate, skew-symmetric bilinear form $Q\colon V \times V \to \mathbb{F}$.

Definition 2.1. *We give a standard construction for sympletic forms. Let H be a Hilbert space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle\colon H \times H \to \mathbb{R}$. The form $\langle \cdot, \cdot \rangle$ extends uniquely to a nondegenerate symmetric complex bilinear form on $H_{\mathbb{C}} = H + iH$,*

$$\beta(u + iv, x + iy) = \langle u, x\rangle - \langle v, y\rangle + i(\langle u, y\rangle + \langle v, x\rangle).$$

Note that $H_{\mathbb{C}}$ is a complex Hilbert space with inner product $\langle \mathbf{a}, \mathbf{b}\rangle = \beta(\mathbf{a}, \overline{\mathbf{b}})$, where $\overline{\mathbf{b}} = u - iv$ if $\mathbf{b} = u + iv$, and that both $\beta(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ restricted to $H \subseteq H_{\mathbb{C}}$ agree with $\langle \cdot, \cdot \rangle$. For $\mathbb{F} = \mathbb{R}$, we set $E = H$ and for $\mathbb{F} = \mathbb{C}$, we set $E = H_{\mathbb{C}}$. Set $V_E := E \oplus E$; we denote members of V_E by column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x, y \in E$. We define the sympletic form $Q := Q_E$ on V_E by

$$Q_E\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) := \beta(x_1, y_2) - \beta(y_1, x_2),$$

where $\beta(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ for $\mathbb{F} = \mathbb{R}$. The pair (V_E, Q_E) is called a standard sympletic space*, real if $\mathbb{F} = \mathbb{R}$ and complex if $\mathbb{F} = \mathbb{C}$.*

Example 2.2. *Let $E = \mathbb{R}$ be the one-dimensional Hilbert space over $\mathbb{R}$. Then $V_E = \mathbb{F} \oplus \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and*

$$Q_E\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) = \det \begin{bmatrix} w_1 & z_1 \\ w_2 & z_2 \end{bmatrix}, \ w_1, w_2, z_1, z_2 \in \mathbb{F}.$$

Example 2.3. *Let $E = \mathbb{R}^n$ with its usual Hilbert space structure. Then $V_E = \mathbb{F}^{2n}$. The sympletic form restricted to the standard basis is given by*

$$Q(e_i, e_j) = \begin{cases} 1, & \text{if } j = i + n; \\ -1, & \text{if } i = j + n; \\ 0, & \text{otherwise.} \end{cases}$$

The sympletic form Q may be expressed as the matrix product

$$Q(x, y) = x^* \cdot J \cdot y$$

where x^* denotes the transpose and J denotes the $2n \times 2n$ matrix given in block form as

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Remark 2.4. *Given any sympletic form Q on a finite dimensional vector space V, there exists a basis $\{\epsilon_1, \ldots, \epsilon_{2n}\}$ such that Q restricted to this basis is given by the formulas in Example 2.3. Thus the sympletic space (V, Q) is isomorphic as a symplectic space to the standard one of Example 2.3 under the isomorphism that carries ϵ_i to e_i.*

Let $(V_E, Q_E = Q)$ be a standard sympletic space, where E is a Hilbert space. Any bounded linear operator $A\colon V_E \to V_E$ then has a block matrix representation of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ where } A_{ij} := \pi_i \circ A \circ \iota_j \colon E \to E,$$

where $\iota_j\colon E \to V_E$ is the natural embedding into the j-th coordinate and $\pi_i\colon V_E \to E$ is projection into the i-th coordinate, for $i, j = 1, 2$. We denote by $\mathrm{End}(V_E)$ resp. $\mathrm{End}(E)$ the set of bounded $\mathbb{F}$-linear operators on V_E resp. E, and by $\mathrm{GL}(V_E)$ resp. $\mathrm{GL}(E)$ those that are invertible. We shall always assume the topology is generated by the operator norm. Note that the operator norm topology on $\mathrm{End}(V_E)$ is the product topology of the operator norm topology for the four block matrix operators in $\mathrm{End}(E)$.

We define an operator $J \in \mathrm{End}(V_E)$ by

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the identity operator on E. Note that

$$J^2 = -I_{V_E},\ J^4 = I_{V_E};$$

in particular, J is invertible.

There are a direct sum non-degenerate symmetric bilinear form $\beta(\cdot,\cdot)$ and a Hilbert space inner product $\langle\cdot,\cdot\rangle$ defined on V_E by

$$\beta(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}) := \beta(u,x) + \beta(v,y), \quad \langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle := \langle u,x\rangle + \langle v,y\rangle.$$

In terms of β, the symplectic form Q is given by $Q(\mathbf{a},\mathbf{b}) = \beta(\mathbf{a}, J\mathbf{b})$.

For a bounded linear transformation A on E or V_E, let A^* denote the unique linear operator such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all x, y in E or V_E respectively. We call A^* the *adjoint* of A. Note that for $\mathbb{F} = \mathbb{C}$, $x, y \in E$,

$$\beta(x, Ay) = \langle x, \overline{Ay}\rangle = \langle x, A\overline{y}\rangle = \langle A^*x, \overline{y}\rangle = \beta(A^*x, y),$$

so A^* is the same as the adjoint with respect to $\beta(\cdot,\cdot)$, and a similar computation is valid in V_E. Observe that for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathrm{End}(V_E)$, we have by straightforward computation

$$\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle = \langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rangle.$$

Thus $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$. It follows that

$$J^* = -J,\ JJ^* = I.$$

We say that A is *symmetric* if $A^* = A$.

We denote by $M^\sharp$ for $M \in \mathrm{End}(V_E)$ the unique linear operator such that $Q(Mx, y) = Q(x, M^\sharp y)$ for all $x, y \in V_E$. Since

$$\begin{aligned} Q(x, M^\sharp y) &= Q(Mx, y) = \beta(Mx, Jy) = \beta(x, M^*Jy) \\ &= \beta(x, JJ^*M^*Jy) = Q(x, J^*M^*Jy), \end{aligned}$$

we conclude that $M^\sharp = J^*M^*J = -JM^*J$. We call $M^\sharp$ the *sympletic conjugate* of M.

For (V_E, Q) a standard sympletic space, we set

$$\mathrm{Sp}(V_E) := \{M \in \mathrm{GL}(V_E) \colon \forall x, y \in V_E,\ Q(Mx, My) = Q(x, y)\}.$$

Suppose that $M \in \mathrm{Sp}(V_E)$. Then $\forall x, y \in V_E$,

$$\beta(x, Jy) = Q(x, y) = Q(Mx, My) = \beta(Mx, JMy) = \beta(x, M^*JMy).$$

Thus $J = M^*JM$, and the argument reverses to yield that $M \in \mathrm{Sp}(V_E)$ if M is invertible and $M^*JM = J$. We conclude that for $M \in \mathrm{GL}(V_E)$,

$$M^*JM = J \Leftrightarrow M \in \mathrm{Sp}(V_E).$$

Proposition 2.5. *Let $M \in \mathrm{GL}(V_E)$. The following are equivalent:*

(1) *$M \in \mathrm{Sp}(V_E)$, i.e., M preserves $Q(\cdot, \cdot)$;*
(2) *$M^*JM = J$;*
(3) *If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then*
  (a) *$A^*C,\ B^*D$ are symmetric;*
  (b) *$A^*D - C^*B = I$.*
(4) *$M^{-1} = M^\sharp$.*

Thus the set $\mathrm{Sp}(V_E)$ is a group.

Proof. The equivalence of (1) and (2) was established in the remarks preceding the proposition. The equivalence of (2) and (3) is a straightforward computation. The implication (2)⇒(4) follows from multiplying both sides of (2) on the left by $-J$, and left multiplying $M^\sharp M = I$ by J gives the reverse implication.

It follows from definition that $\mathrm{Sp}(V_E)$ is closed under composition and from (4) that it is closed under inversion (since $M^{\sharp\sharp} = M$); hence it is a group. □

Corollary 2.6. *The set $\mathrm{Sp}(V_E)$ is closed under taking adjoint. Hence $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E)$ if and only if*

(1) *$AB^*,\ CD^*$ are symmetric;*
(2) *$AD^* - BC^* = I$.*

Proof. Let $M \in \mathrm{Sp}(V_E)$. By (4) of the preceding proposition, we have

$$(M^*)^{-1} = (M^{-1})^* = (M^\sharp)^* = (J^*M^*J)^* = J^*M^{**}J = (M^*)^\sharp,$$

and thus $M^* \in \mathrm{Sp}(V_E)$. The remaining assertion follows from applying the preceding proposition to M^*. □

Recall that the Lie algebra $\mathfrak{sp}(V_E)$ consists of all $X \in \mathrm{End}(V_E)$ such that $\exp(tX) \in \mathrm{Sp}(V_E)$ for all $t \in \mathbb{R}$.

Proposition 2.7. *Let $X \in \mathrm{End}(V_E)$. The following are equivalent:*

(1) $X \in \mathfrak{sp}(V_E)$,
(2) $X^*J + JX = 0$;
(3) *If* $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *then*
 (a) B *and* C *are symmetric;*
 (b) $D = -A^*$.

Proof. (1) ⇔ (2): Suppose that $X \in \mathfrak{sp}(V_E)$. Then by (2) of Proposition 2.5 $e^{tX^*}Je^{tX} = J$ for all $t \in \mathbb{R}$. Differentiating with respect to t and evaluating at $t = 0$ yields the desired result. Conversely if (2) is satisfied, then the function $t \mapsto e^{tX^*}Je^{tX}$ has derivative (with respect to t) $e^{tX^*}(X^*J+JX)e^{tX} = 0$, and hence is a constant function. Evaluating at $t = 0$ establishes that the constant is J.

(2) ⇔ (3): This is a straightforward computation using the block operator representation. □

3. The Riccati Equation

Let E be a Hilbert space and let $V_E = E \oplus E$ as in the previous section. We consider the control system given by the *basic group control equation* (BGCE) on $\mathrm{Sp}(V_E)$:

$$\dot{g}(t) = u(t)g(t), \qquad \text{(BGCE)}$$

where $u : \mathbb{I} \to \mathfrak{sp}(V_E)$, $\mathbb{I}$ a (finite or infinite) subinterval of $\mathbb{R}$, is called a *steering* or *control* function. In the case that E is finite dimensional, we assume that $u(\cdot)$ belongs to the class of measurable functions from $\mathbb{I}$ into $\mathfrak{sp}(V_E)$ which are locally bounded, that is, bounded on every finite subinterval, and in the case of general E we assume that $u(\cdot)$ is a regulated function, that is, a function that on each finite subinterval of its domain is a uniform limit of piecewise constant functions. A solution of (BGCE), called a *trajectory*, is an absolutely continuous function $x(\cdot)$ from $\mathbb{I}$ into G such that the equation (BGCE) holds a.e., where a.e. means on the complement of a set of measure 0 in the finite dimensional setting and the complement of a countable set otherwise. Control systems such as the one just described are called *right invariant*, since right translates of solutions of (BGCE) are again solutions. Using the homogenity of $\mathrm{Sp}(V_E)$, one readily obtains that global solutions on all of $\mathbb{I}$ exist whenever local solutions exist. Thus global solutions always exist in the settings we are considering (see [8, Section 3] and [4, Section IV.5]). The solution for initial condition $g(0) = \mathrm{id}_{V(E)}$ is called the *fundamental solution* of the basic group control equation and denoted $\Phi(t)$. By right invariance the general solution to (BGCE) with initial condition $g(t_0) = g_0$ is then given by $g(t) = \Phi(t)(\Phi(t_0))^{-1}g_0$.

We turn now to Ricatti equations.

Definition 3.1. *An* (operator) Riccati equation *is a differential equation on the space* $\mathrm{Sym}(E)$ *of bounded symmetric operators on* E *of the form*

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \;\; K(t_0) = K_0, \qquad \text{(R)}$$

where $R(t), S(t), K_0$ *are all in* $\mathrm{Sym}(E)$.

There is a close connection between the basic group control equation and the Riccati equation.

Lemma 3.2. *Suppose that* $g(\cdot)$ *is a solution of the following* (*BGCE*) *on an interval* $\mathbb{I}$*:*

$$\dot{g}(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A^*(t) \end{bmatrix} \begin{bmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{bmatrix}, \;\; R(t), S(t) \in \mathrm{Sym}(E).$$

If g_{22} *is invertible for all* $t \in \mathbb{I}$*, then* $K(t) := g_{12}(t)(g_{22}(t))^{-1}$ *satisfies*

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t),$$

on $\mathbb{I}$*. Furthermore, if* $g(t_0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}$ *for some* $t_0 \in \mathbb{I}$*, then* $K(t_0) = K_0$*.*

Proof. Using the product rule and the power rule for inverses and the equality of the second columns in the basic group control equation, we obtain

$$\begin{aligned}\dot{K} &= \dot{g}_{12}(g_{22})^{-1} - g_{12}g_{22}^{-1}\dot{g}_{22}g_{22}^{-1} \\ &= (Ag_{12} + Rg_{22})g_{22}^{-1} - K(Sg_{12} - A^*g_{22})g_{22}^{-1} \\ &= AK + R - KSK + KA^*.\end{aligned}$$

The last assertion is immediate. □

Corollary 3.3. *Local solutions exist for the Riccati equation* (*R*).

Proof. Global solutions exist for the basic group control equation (BGCE) with initial condition $g(t_0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}$ and the $g_{22}(t)$-entry will be invertible in some neighborhood of t_0. Now apply the previous theorem. □

4. The Spaces Λ and $\mathcal{M}$

We fix the Hilbert space E and define

$$\Lambda := \left\{ \begin{bmatrix} B \\ D \end{bmatrix} : \exists A, C \in \mathrm{End}(E) \text{ such that } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) \right\}.$$

We also consider the lower block triangular subgroup $\mathbf{P}$ of $\mathrm{Sp}(V_E)$ given by

$$\mathbf{P} := \left\{ \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : A, C, D \in \mathrm{End}(E) \right\}.$$

We note from Proposition 2.5 that such a lower triangular block matrix is in $\mathrm{Sp}(V_E)$ if and only if $A^* = D^{-1}$ and $A^*C = D^{-1}C$ is symmetric.

Proposition 4.1. *Let* $\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} \in \Lambda$. *The following are equivalent:*

(1) *There exist* $M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$, $M_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \in \mathrm{Sp}(V_E)$ *such that* $M_1\mathbf{P} = M_2\mathbf{P}$.

(2) *For all* $M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$, $M_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \in \mathrm{Sp}(V_E)$, *we have* $M_1\mathbf{P} = M_2\mathbf{P}$.

(3) *There exists* $Q \in \mathrm{GL}(E)$ *such that* $B_1Q = B_2$ *and* $D_1Q = D_2$.

Proof. (1)⇒(3): This implication follows directly from the fact $M_2 \in M_1\mathbf{P}$; the matrix Q is the lower right-hand entry of the matrix $P \in \mathbf{P}$ such that $M_2 = M_1 P$.

(3)⇒(1): Assume (3). Pick $\begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} \in \mathrm{Sp}(V_E)$. Then

$$\begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} \begin{bmatrix} (Q^*)^{-1} & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} A(Q^*)^{-1} & B_1 Q = B_2 \\ C(Q^*)^{-1} & D_1 Q = D_2 \end{bmatrix}.$$

Note that the right-hand side is in $\mathrm{Sp}(V_E)$, since the left-hand factors are. Also the second factor in the left-hand side is in $\mathbf{P}$, so (1) follows.

(1)⇔(2): The implication from right to left is trivial. Assume (1). It suffices to show that if there is another matrix $M = \begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} \in \mathrm{Sp}(V_E)$, then $M_1\mathbf{P} = M\mathbf{P}$. Since M_1 and M have the same second column, when left multiplied by ${M_1}^{-1}$ the second columns must remain equal, i.e., $\hat{M} := {M_1}^{-1}M$ must be of the form $\begin{bmatrix} * & 0 \\ * & I \end{bmatrix}$. Since $\hat{M} \in \mathrm{Sp}(V_E)$, it follows that $\hat{M} \in \mathbf{P}$, since it is block lower triangular. Thus $M_1\mathbf{P} = M\mathbf{P}$. □

Definition 4.2. *We define* $\begin{bmatrix} B_1 \\ D_1 \end{bmatrix} \sim \begin{bmatrix} B_2 \\ D_2 \end{bmatrix}$ *in case the equivalent conditions of Proposition 4.1 hold. The relation* $\sim$ *is an equivalence relation (from part (2) or (3)) and the quotient space* $\Lambda/\sim$ *is denoted* $\mathcal{M}$. *We denote the equivalence class of* $\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}$ *by* $\begin{pmatrix} B_1 \\ D_1 \end{pmatrix}$.

There exists a natural projection $\pi\colon \mathrm{Sp}(V_E) \to \Lambda$ which sends a matrix to its second column. Let $\rho\colon \Lambda \to \mathcal{M}$ be the natural projection from Λ to $\mathcal{M}$ which sends a column to its $\sim$-equivalence class. We endow $\mathcal{M}$ with the quotient topology from $\rho \circ \pi$.

Corollary 4.3. *Consider* $\psi := \rho\circ\pi\colon \mathrm{Sp}(V_E) \to \mathcal{M}$. *Then* $\psi(M_1) = \psi(M_2)$ *if and only if* $M_1\mathbf{P} = M_2\mathbf{P}$. *Thus the left transformation group* $(\mathrm{Sp}(V_E), \mathcal{M})$, *where the action is given by left block matrix multiplication by any representative of a* $\sim$*-equivalence class, is topologically conjugate to the coset transformation group* $(\mathrm{Sp}(V_E), \mathrm{Sp}(V_E)/\mathbf{P})$ *and the mapping* ψ *is open.*

Proof. The first assertion follows readily from Proposition 4.1, and the second assertion follows readily from the first. It is standard that the quotient mapping onto a homogeneous space is an open mapping. □

The preceding corollary allows us to freely identify $\mathcal{M}$ and the homogenous space $\mathrm{Sp}(V_E)/\mathbf{P}$. In the finite dimensional setting $\mathbf{P}$ is a parabolic subgroup and the homogenous space is a flag manifold of $\mathrm{Sp}(V_E)$.

We say that a point $\begin{pmatrix} B \\ D \end{pmatrix} \in \mathcal{M}$ is *finite* if D is invertible (note from part (3) of Proposition 4.1 this invertibility is independent of which representative is chosen). In this case we may rewrite the point as $\begin{pmatrix} BD^{-1} \\ I \end{pmatrix}$. Since for appropriate A, C,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D^* & 0 \\ 0 & D^{-1} \end{bmatrix} = \begin{bmatrix} * & BD^{-1} \\ * & I \end{bmatrix} \in \mathrm{Sp}(V_E),$$

we conclude $(BD^{-1})^* I$ is symmetric, and hence BD^{-1} is symmetric. Conversely if $E \in \mathrm{End}(E)$ is symmetric, then $\begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \in \mathrm{Sp}(V_E)$, and thus $\begin{pmatrix} E \\ I \end{pmatrix} \in \mathcal{M}$.

Proposition 4.4. *The correspondence* $A \leftrightarrow \begin{pmatrix} A \\ I \end{pmatrix}$ *is a homeomorphism between the set* $\mathrm{Sym}(E)$ *of symmetric operators in* $\mathrm{End}(E)$ *and the open set* $\mathcal{M}_0$ *of finite points in* $\mathcal{M}$.

Proof. Since by (3) of Proposition 4.1 we can represent each member of $\mathcal{M}$ in at most one way with bottom entry I, we have from the preceding discussion that the correspondence is a bijection.

Since the operator norm topology for the block operator matrices agrees with the product topology from the operator norm topologies in each block, we conclude that the mapping

$$\beta : \mathrm{Sym}(E) \to \mathcal{M} \text{ defined by } A \mapsto \begin{bmatrix} A \\ I \end{bmatrix} \mapsto \begin{pmatrix} A \\ I \end{pmatrix}$$

is continuous.

Conversely consider the open subset $U \subseteq \mathrm{Sp}(V_E)$ of elements such that the $(2,2)$-block entry D is invertible. The open set U is the inverse image of set of finite points of $\mathcal{M}$. Thus the set $\mathcal{M}_0$ of finite points is open in $\mathcal{M}$, since the quotient map is open. The map

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto BD^{-1} : U \to \mathrm{Sym}(E)$$

is continuous, and induces $\begin{pmatrix} B \\ D \end{pmatrix} \mapsto BD^{-1}$ on the quotient space $\mathcal{M}$, and thus the latter map is also continuous (note that the maps do go into $\mathrm{Sym}(E)$ by the paragraph preceding the proposition). Thus the correspondence is a homeomorphism. □

Remark 4.5. Our presentation of the manifold $\mathcal{M}$ is nonstandard. Typically, at least in the finite dimensional setting, one considers maximal isotropic subspaces of V_E, sometimes called Lagrangian subspaces or polarizations. The "horizontal" subspace $E_H = E \oplus \{0\}$ and the "vertical" subspace $E_V = \{0\} \oplus E$ are examples of such. If in our context we define a polarization of V_E to be an image of E_V under a member of $\mathrm{Sp}(V_E)$, then we can identify the members of $\mathcal{M}$ with the polarizations of V_E via the correspondence

$$\begin{pmatrix} B \\ D \end{pmatrix} \leftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} E_V, \ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E).$$

Another way of saying this is that we associate with $\begin{pmatrix} B \\ D \end{pmatrix}$ the column space of $\begin{bmatrix} B \\ D \end{bmatrix}$.

5. Extended Solutions of Riccati Equations

The results of the preceding section allow us to extend the solution of a Ricatti equation by considering it to be a differential equation on the larger $\mathcal{M}$ with $\mathrm{Sym}(E)$ embedded as the set of finite points as outlined in the previous section.

Consider on E the Riccati equation

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \ K(t_0) = K_0, \qquad \text{(R)}$$

where t varies over some interval $\mathbb{I}$ containing t_0. (We recall our standing hypothesis that coefficient functions are locally bounded and measurable in the finite dimensional case and regulated otherwise.) As we have seen in Section 3, we can obtain a solution to the Riccati equation from the solution of the basic group control equation

$$\dot{g}(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A^*(t) \end{bmatrix} \begin{bmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{bmatrix}, \; g(t_0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}$$

by setting $K(t) = g_{12}(t)(g_{22}(t))^{-1}$ on any interval containing t_0 where $g_{22}(t)$ is invertible.

Note that by uniqueness of solutions we have $g(t) = k(t)$, where we define

$$k(t) := \Phi(t)(\Phi(t_0))^{-1} \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix},$$

since both satisfy (BGCE) and have the same initial point at t_0.

Suppose that on the interval $\mathbb{I}$ we define a function $\tilde{K}(\cdot)$ on $\mathcal{M}$ by

$$\begin{aligned} \tilde{K}(t) &= g(t) \begin{pmatrix} 0 \\ I \end{pmatrix} = k(t) \begin{pmatrix} 0 \\ I \end{pmatrix} \\ &= \Phi(t)(\Phi(t_0))^{-1} \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = \Phi(t)(\Phi(t_0))^{-1} \begin{pmatrix} K_0 \\ I \end{pmatrix}. \end{aligned} \tag{ES}$$

We observe on any interval where $g_{22}(t)$ is invertible that

$$\tilde{K}(t) = \begin{pmatrix} g_{12}(t) \\ g_{22}(t) \end{pmatrix} = \begin{pmatrix} g_{12}(t)(g_{22}(t))^{-1} \\ I \end{pmatrix} = \begin{pmatrix} K(t) \\ I \end{pmatrix},$$

where the last equality follows from Lemma 3.2. The last expression also agrees with the embedded image in $\mathcal{M}$ of the solution $K(t)$. The function $\tilde{K}(\cdot)$ on $\mathbb{I}$ is called the *extended solution of the Riccati equation.*

Consider the maximal interval around t_0 for which $g_{22}(t)$ is invertible. This inteval is open since g_{22} is continuous and $\mathcal{M}_0$ is open in $\mathcal{M}$. By uniqueness of solutions any solution $K_1(\cdot)$ of the Riccati equation (R) and $\tilde{K}(\cdot)$ must agree on this interval (we are viewing $\mathrm{Sym}(E)$ as embedded in $\mathcal{M}$). If $K_1(\cdot)$ admitted a solution at the endpoint t_1, then by continuity $K_1(t_1) = \tilde{K}(t_1)$, which is impossible since one is a finite point and the other is not. We have thus established the following

Proposition 5.1. *The Riccati equation (R) admits an extended solution throughout the interval* $\mathbb{I}$ *on which it is defined. The maximal interval on which (R) admits a solution is the largest interval containing* t_0 *such that the extended solution is finite.*

The basic group control equation (BGCE) pushes forward to a control system on the manifold $\mathcal{M}$ in such a way that the restriction to $\mathrm{Sym}(E)$, the space of finite points, agrees with the Riccati equation. We indicate briefly this "push-forward" construction. Let M be a smooth ($=C^\infty$) manifold and suppose that $\Psi : G\times M\to M$ is a smooth action of a Lie group G on M. In our case $G=\mathrm{Sp}(V_E)$ and $M=\mathcal{M}$ endowed with the appropriate smooth structure to make the action of $\mathrm{Sp}(V_E)$ on $\mathcal{M}$ smooth. (This smooth structure arises by taking the inverse of the embedding of $\mathrm{Sym}(E)$ into $\mathcal{M}$ and its translates by members of $\mathrm{Sp}(V_E)$ acting on $\mathcal{M}$ as an atlas of charts.) We typically denote $\Psi(g,x)$ by gx or $g.x$. Let $V^\infty(M)$ denote the Lie algebra of smooth vector fields on M. For $x\in M$, the smooth mapping $\Psi_x : G\to M$ given by $\Psi_x(g)=g.x$ has derivative at e, $d\Psi_x : T_eG\to T_xM$; alternatively $d\Psi_x(v)$, $v\in T_eG$, is given by $v\mapsto \dot\alpha(0)$, where $\alpha:\mathbb{R}\to M$ is defined by $\alpha(t)=\exp(tX).x$, where $\exp:\mathfrak{g}\to G$ is the exponential mapping and $X(e)=v$. The mappings $d\Psi_x$ give rise to a Lie algebra homomorphism $d\Psi:\mathfrak{g}\to V^\infty(M)$ given by $d\Psi(X)(x)=d\Psi_x(X(e))$ (note that the appropriate match-up to obtain a Lie algebra homomorphism is right invariant vector fields with left actions). We denote the vector field $d\Psi(X)$ by $\vec{X}$. We consider the *basic manifold control equation* (BMCE) on M given by the control differential equation

$$\dot x(t)=\vec{u}(t)\,(x(t))\,,$$

where $u(\cdot):\mathbb{I}\to\mathfrak{g}$ is locally bounded and $\vec{u}(t)=d\Psi\,(u(t))$.

Proposition 5.2. *The solution to the (BMCE)*

$$\dot x(t)=\vec{u}(t)\,(x(t))\,,\quad x(t_0)=x_0$$

on M is given by $x(t)=\Phi(t)(\Phi(t_0))^{-1}.x_0$. The basic control differential equation on M has a global solution for any initial value.

Proof. The first assertion follows from

$$\begin{aligned}\dot x(t) &= d\Psi_{x_0}\dot\Phi(t)(\Phi(t_0))^{-1}=d\Psi_{x_0}\left(u(t)\Phi(t)(\Phi(t_0))^{-1}\right)=d\Psi_{x_0}\circ d\rho_{\Phi(t)(\Phi(t_0))^{-1}}(u(t)e)\\ &= d\Psi_{\Phi(t)(\Phi(t_0))^{-1}.x_0}(u(t)e)=\vec{u}\,(x(t))\,,\end{aligned}$$

where $\rho_g(h)=hg$ is right translation in G. That global solutions exist now follows from the corresponding assertion for the (BGCE). The last assertion follows readily from the first. ∎

Remark 5.3. In the case $G = \mathrm{Sp}(V_E)$ and $M = \mathcal{M}$ that we are considering, we note from equation (ES) above that

$$\tilde{K}(t) = \Phi(t)(\Phi(t_0))^{-1}\begin{pmatrix} K_0 \\ I \end{pmatrix},$$

which is the solution of (BMCE) for initial condition $\begin{pmatrix} K_0 \\ I \end{pmatrix}$ at time t_0. Hence the extended solution of the Riccati equation is the solution of (BGCE) pushed forward to (BMCE).

6. The Symplectic Semigroup

Let (V_E, Q) be a standard sympletic space constructed from a real Hilbert space H. A bounded symmetric operator A on E is *positive semidefinite* if $\langle x, Ax\rangle \geq 0$ for all $x \in E \setminus \{0\}$. We denote by $\mathcal{P}$ (resp. $\mathcal{P}_0$) all positive semidefinite (resp. positive semidefinite invertible) bounded operators on E. We use freely the standard fact from operator theory that a positive semidefinite operator has a unique positive semidefinite square root.

Lemma 6.1. *If $P, Q \in \mathcal{P}$ then $I + PQ$ is invertible. If $P \in \mathcal{P}_0$ and $Q \in \mathcal{P}$, then $P + Q \in \mathcal{P}_0$.*

Proof. We first show that $I + PQ$ is injective. For if $(I + PQ)(x) = 0$, then

$$0 = \langle Q(x), (I + PQ)(x)\rangle = \langle Q(x), x\rangle + \langle Q(x), PQ(x)\rangle.$$

Since both latter terms are non-negative by hypothesis, we have that $0 = \langle Qx, x\rangle = \langle Q^{1/2}x, Q^{1/2}x\rangle$, and thus that $Q^{1/2}(x) = 0$. It follows that $0 = (I + PQ)(x) = x + PQ^{1/2}(Q^{1/2}x) = x$, and thus $I + PQ$ is injective.

The same argument may be applied to the adjoint $I + QP$ to conclude that it is also injective, and hence its adjoint $I + PQ$ has dense image.

Suppose that $(I + PQ)(x_n) \to 0$. We claim that $x_n \to 0$. For if not, then we obtain a subsequence, again denoted x_n, such that x_n is bounded away from 0, i.e., there exists $\beta > 0$ such that $\beta < \|x_n\|$ for all n. Then $u_n := x_n/\|x_n\| = (\epsilon_n/\beta)x_n$ for some $0 < \epsilon_n < 1$. Since $(1/\beta)(I + PQ)(x_n) \to 0$, it follows that $(I + PQ)(u_n) \to 0$. Since

$\|Q(u_n)\| \le \|Q\|$ for all n, we have

$$\langle Q(u_n), u_n\rangle + \langle Q(u_n), PQ(u_n)\rangle = \langle Q(u_n), (I+PQ)(u_n)\rangle \to 0.$$

Since both of the left-hand terms are non-negative, we have $\langle Q^{1/2}(u_n), Q^{1/2}(u_n)\rangle = \langle Q(u_n), u_n\rangle \to 0$, and thus that $Q^{1/2}(u_n) \to 0$. By continuity of $PQ^{1/2}$, it follows that $PQ(u_n) \to 0$, and thus that $\|(I+PQ)(u_n)\| \to 1$, a contradiction to $(I+PQ)(u_n) \to 0$.

We conclude by showing that $I + PQ$ is surjective. Let $y \in E$. Then there exists $x_n \in E$ such that $y_n := (I + PQ)(x_n) \to y$, since $I + PQ$ has dense image. Then the double indexed sequence $y_n - y_m = (I + PQ)(x_n - x_m) \to 0$ as $m, n \to \infty$. It then follows from the preceding paragraph that the double indexed sequence $x_n - x_m \to 0$ as $m, n \to \infty$, i.e., the sequence $\{x_n\}$ is Cauchy. Let x be its limit. By continuity $(I+PQ)(x) = y$. Thus $I+PQ$ is surjective. By the Banach Open Mapping Theorem, it is open, so the inverse is a bounded linear operator.

The last assertion now follows easily by observing that $P + Q = P(I + P^{-1}Q)$. □

Remark 6.2. *Note that the preceding proof simplifies considerably in the finite dimensional case. Indeed it follows from the injectivity of $I + PQ$ that it is invertible.*

We define four subsets:

$$\mathcal{S} = \left\{\begin{bmatrix} A & B \\ C & D\end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible},\ B^*D \in \mathcal{P},\ CD^* \in \mathcal{P}\right\},$$

$$\mathcal{S}_1 = \left\{\begin{bmatrix} A & B \\ C & D\end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible},\ B^*D \in \mathcal{P}_0,\ CD^* \in \mathcal{P}\right\},$$

$$\mathcal{S}_2 = \left\{\begin{bmatrix} A & B \\ C & D\end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible},\ B^*D \in \mathcal{P},\ CD^* \in \mathcal{P}_0\right\},$$

$$\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2.$$

Remark 6.3. *Note that $\mathcal{S}_2$ is the adjoint dual of $\mathcal{S}_1$ and that $\mathcal{S}$ is self-dual, i.e., $\mathcal{S}$ is closed under adjoints.*

Members of $\mathcal{S}$ are sometimes called *Hamiltonian* operators of $\mathrm{Sp}(V_E)$.

We define

$$\Gamma^U = \left\{\begin{bmatrix} I & B \\ 0 & I\end{bmatrix} : B \in \mathcal{P}\right\},\ \Gamma^U_0 = \left\{\begin{bmatrix} I & B \\ 0 & I\end{bmatrix} : B \in \mathcal{P}_0\right\},$$

$$\Gamma^L = \left\{ \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} : C \in \mathcal{P} \right\},\ \Gamma_0^L = \left\{ \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} : C \in \mathcal{P}_0 \right\}.$$

We further define a group H of block diagonal matrices by

$$H = \left\{ \begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix} : A \in \mathrm{GL}(E) \right\}.$$

The following lemma is straightforward.

Lemma 6.4. *All four of the sets Γ^U, Γ^L, Γ_0^U and Γ_0^L are semigroups under composition, the first two are closed, and Γ_0^U (resp. Γ_0^L) is a semigroup ideal in Γ^U (resp. Γ^L). The semigroup Γ^U resp. Γ_0^U (resp. Γ^L resp. Γ_0^L) consists of all unipotent block upper (resp. lower) triangular operators contained in $\mathcal{S}$ resp. $\mathcal{S}_1$ (resp. $\mathcal{S}$, resp. $\mathcal{S}_2$). The group H is closed in $\mathrm{GL}(V_E)$ and consists of all block diagonal matrices in $\mathrm{Sp}(V_E)$. Furthermore, each of the four semigroups Γ^U, Γ^L, Γ_0^U, and Γ_0^L is invariant under conjugation by members of H.*

Lemma 6.5. *We have that $\mathcal{S} = \Gamma^U H \Gamma^L$, $\mathcal{S}_1 = \Gamma_0^U H \Gamma^L$, $\mathcal{S}_2 = \Gamma^U H \Gamma_0^L$, and $\mathcal{S}_0 = \Gamma_0^U H \Gamma_0^L$. Furthermore these "triple decompositions" are unique. The multiplication mapping from $\Gamma^U \times H \times \Gamma^L$ to $\mathcal{S}$ is a homeomorphism.*

Proof. Each member of $\mathcal{S}$ admits a triple decomposition of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (D^{-1})^* & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}. \tag{1}$$

The triple decomposition follows from direct multiplication (applying the equations $A^*D - C^*D = I$ and $B^*D = D^*B$ to see that the (1,1)-entry is A). Note further that if $B^*D = D^*B \in \mathcal{P}$ (resp. $\mathcal{P}_0$), then $BD^{-1} = (D^{-1})^*D^*BD^{-1} \in \mathcal{P}$ (resp. $\mathcal{P}_0$), and hence the first factor in the triple decomposition is in Γ^U (resp. Γ_0^U). Similar reasoning applies to the third factor after noting $D^{-1}C = D^{-1}CD^*(D^{-1})^*$.

Conversely consider a product

$$\begin{bmatrix} D^{-1} + BD^*C & BD^* \\ D^*C & D^* \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} D^{-1} & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \in \Gamma^U H \Gamma^L.$$

Then the (2,2)-entry in the product is precisely D^* and the middle block diagonal matrix in the factorization is determined. Multiplying the $(1,2)$-entry of the product on the right by $(D^*)^{-1}$ gives B and the $(2,1)$-entry on the left by $(D^*)^{-1}$ gives C. Hence the triple factorization is uniquely determined. Finally note that $(BD^*)^*D^* = DB^*D^*$ is positive semidefinite since B is (since the first block matrix is in Γ^U). Also $(D^*C)(D^*)^* = D^*CD$, which is positive semidefinite since C is. Thus the product block matrix satisfies the conditions to be in $\mathcal{S}$. Note further that DB^*D^* resp. D^*CD is invertible if B resp. C is, and thus the decomposition holds also in $\mathcal{S}_i$, $i = 0, 1, 2$.

In regard to the last statement, we have seen that the mapping is a bijection, it is continuous since multiplication (i.e., composition) is, and from the first display in the proof we see that the inverse factorization is also continuous on $\mathcal{S}$. □

Related triple decompositions in the finite dimensional setting have been obtained by Wojtkowski [14] for the real sympletic group, by Koufany [7] in the setting of euclidean Jordan algebras, and by the authors in the setting of Lie algebras of Cayley type [9].

Remark 6.6. *Analogous triple decompositions occur for the larger set of sympletic block matrices for which the* $(2,2)$*-block* D *is invertible. For this set of matrices, one has unique triple decompositions in the set product* $N^U H N^L$ *where* N^U *resp.* N^L *denotes the group of upper resp. lower block unipotent matrix operators. The preceding proof adapts directly to this case.*

The following semigroup property appears in the finite dimensional setting in Bougerol [1] and in Wojtkowski [14] and [15].

Theorem 6.7. *We have that* $\mathcal{S}$ *is a semigroup. Furthermore,* $\mathcal{S}\mathcal{S}_i\mathcal{S} \subseteq \mathcal{S}_i$ *for* $i = 0, 1, 2$, *i.e., each* $\mathcal{S}_i$ *is a semigroup ideal.*

Proof. Let $s_1 = u_1h_1l_1$ and $s_2 = u_2h_2l_2$ be the triple decompositions for $s_1, s_2 \in \mathcal{S}$. Suppose that $l_1u_2 = u_3h_3l_3 \in \Gamma^U H \Gamma^L$. That

$$s_1s_2 = u_1h_1l_1u_2h_2l_2 = u_1h_1u_3h_3l_3h_2l_2 = [u_1(h_1u_3h_1^{-1})](h_1h_3h_2)[(h_2^{-1}l_3h_2)l_2]$$

is in $\Gamma^U H \Gamma^L$ then follows from Lemma 6.4. We observe that indeed

$$l_1u_2 = \begin{bmatrix} I & 0 \\ C_1 & I \end{bmatrix} \begin{bmatrix} I & B_2 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & B_2 \\ C_1 & I + C_1B_2 \end{bmatrix},$$

and that the $(2,2)$-entry is invertible by Lemma 6.1. We further have that $B_2^*(I + C_1B_2) = B_2^* + B_2^*C_1B_2$ is positive semidefinite (and is in $\mathcal{P}_0$ if $B_2 \in \mathcal{P}_0$ by Lemma

6.1) and $C_1(I + C_1B_2)^* = C_1 + C_1B_2C_1^*$ is positive semidefinite since C_1 and B_2 are (and is in $\mathcal{P}_0$ if C_1 is). Thus l_1u_2 has the desired triple decomposition $u_3h_3l_3$ and $\mathcal{S}$ is a semigroup by Lemma 6.5. The assertion that the $\mathcal{S}_i$ are ideals now follows readily from Lemmas 6.4 and 6.5 in a similar fashion. □

Definition 6.8. *The semigroup $\mathcal{S}$ of the preceding theorem is called the* sympletic semigroup.

Corollary 6.9. *The sympletic semigroup can be alternatively characterized as*

$$\mathcal{S} = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : A \text{ is invertible, } C^*A \in \mathcal{P},\ BA^* \in \mathcal{P} \right\}.$$

Proof. Let $\mathcal{S}'$ denote the set defined on the righthand side of the equation in the statement of this corollary. We observe that

$$\Delta \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Delta = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \text{ for } \Delta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

The inner automorphism $M \mapsto \Delta M \Delta : \mathrm{GL}(V_E) \to \mathrm{GL}(V_E)$ carries $\mathrm{Sp}(V_E)$ onto itself (check, for example, that it preserves condition (3) of Proposition 2.5), interchanges the semigroups Γ^U and Γ^L, carries the group H to itself, and interchanges the semigroup $\mathcal{S}$ and the set $\mathcal{S}'$. Thus $\mathcal{S}'$ is a semigroup and

$$\mathcal{S}' = \Gamma^L H \Gamma^U \subseteq \mathcal{S}\mathcal{S}\mathcal{S} = \mathcal{S}.$$

Dually $\mathcal{S} \subseteq \mathcal{S}'$. □

7. Fractional Transformations

If $M \in \mathrm{Sp}(V_E)$ and $x, Mx \in \mathcal{M}_0$, the set of finite points, then

$$Mx = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} X \\ I \end{pmatrix} = \begin{pmatrix} AX + B \\ CX + D \end{pmatrix} = \begin{pmatrix} (AX + B)(CX + D)^{-1} \\ I \end{pmatrix}.$$

Identifying $X \in \mathrm{Sym}(E)$ with $\begin{pmatrix} X \\ I \end{pmatrix}$, we have that $MX = (AX + B)(CX + D)^{-1}$, as long as MX is finite.

Proposition 7.1. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E)$. *Then identifying finite points of* $\mathcal{M}$ *with symmetric operators and restricting* M *to the set of finite points whose image under* M *is again finite, we have that* M *acts on this set as the fractional transformation*

$$Z \mapsto (AZ+B)(CZ+D)^{-1}.$$

Members of $\mathcal{S}$ *carry the set* $\mathcal{P}$ *resp.* $\mathcal{P}_0$ *into* $\mathcal{P}$ *resp.* $\mathcal{P}_0$ *via such fractional transformations.*

Proof. We have already observed the first statement. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathcal{S}$ and Z in $\mathcal{P}$ resp. $\mathcal{P}_0$, we have the product

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} * & AZ+B \\ * & CZ+D \end{bmatrix}$$

The right-hand product is in $\mathcal{S}$ resp. $\mathcal{S}_1$ by Theorem 6.7. Thus $(AZ+B)^*(CZ+D) = (CZ+D)^*(AZ+B)$ is in $\mathcal{P}$ resp. $\mathcal{P}_0$ by definition of $\mathcal{S}$ and $\mathcal{S}_1$. Since the right-hand matrix product is in $\mathcal{S}$, we have that $CZ+D$ is invertible. Hence

$$(AZ+B)(CZ+D)^{-1} = ((CZ+D)^{-1})^*[(CZ+D)^*(AZ+B)](CZ+D)^{-1}$$

is in $\mathcal{P}$ resp. $\mathcal{P}_0$. □

Proposition 7.2. *The sympletic semigroup* $\mathcal{S}$ *is closed in* $\mathrm{Sp}(V_E)$.

Proof. Let $\overline{\mathcal{S}}$ denote the closure of $\mathcal{S}$ in $\mathrm{Sp}(V_E)$. By continuity of multiplication $\overline{\mathcal{S}}$ is again a subsemigroup. For $M \in \overline{\mathcal{S}}$, let

$$M_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \to M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \text{ where } M_n \in \mathcal{S} \text{ for all } n.$$

Since $B_n^* D_n \to B^*D$ and the set $\mathcal{P}$ of positive semidefinite operators is closed in $\mathrm{End}(E)$, we conclude that $B^*D \geq 0$. Similarly $CD^* \geq 0$ and the dual conditions $C^*A \geq 0$ and $BA^* \geq 0$ hold.

It is standard that $\mathcal{P}_0$ is open in $\mathrm{Sym}(E) = \mathcal{M}_0$ (we prove this later in Lemma 9.2) and hence in $\mathcal{M}$, since $\mathcal{M}_0$ is open in $\mathcal{M}$. By continuity of the action of $\mathrm{Sp}(V_E)$ on

$\mathcal{M}$, we conclude for $M \in \overline{\mathcal{S}}$ that $M(\mathcal{P}_0) \subseteq \overline{\mathcal{P}}_0$, the closure being taken in $\mathcal{M}$. Since $M(\mathcal{P}_0)$ is open, we conclude that there exists $P > 0$ such that

$$M\begin{pmatrix} P \\ I \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{pmatrix} P \\ I \end{pmatrix} = \begin{pmatrix} AP+B \\ CP+D \end{pmatrix} \in \mathcal{P}_0.$$

It follows that $CP + D$ is invertible. We then have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I & P \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AP+B \\ C & CP+D \end{bmatrix}.$$

The final product is in $\overline{\mathcal{S}}$ since it is a semigroup. Since the positivity conditions hold for any member of $\overline{\mathcal{S}}$ and since the $(2, 2)$-entry is invertible, we conclude that the product is actually in $\mathcal{S}$. But then by the dual conditions of Corollary 6.9, we conclude that the $(1, 1)$-entry A is invertible. If we now apply Corollary 6.9 to M, we conclude that $M \in \mathcal{S}$. □

8. Global Riccati Solutions Via Semigroup Theory

Lie's Fundamental Theorems, which relate Lie groups and Lie algebras, have been extended to Lie semigroups and their tangent objects. For a closed subsemigroup S of a Lie group G, we set

$$\mathfrak{L}(S) := \{X \in \mathfrak{g} : \exp(tX) \in S \text{ for all } t \geq 0\}.$$

It follows directly from the Trotter Product Formula that $\mathfrak{L}(S)$ is a closed convex cone (see [4] or [5]); it is usually referred to as the *Lie wedge* of S, since it is typically not a pointed cone. The semigroup S is said to be *infinitesimally generated* if it is the closure of the semigroup generated by $\exp(\mathfrak{L}(S))$.

Proposition 8.1. *The symplectic semigroup $\mathcal{S}$ has Lie wedge*

$$\mathfrak{L}(\mathcal{S}) = \left\{\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : B, C \geq 0\right\}.$$

Proof. We initially set $\mathcal{W}$ equal to the righthand side of the equation in the statement of the proposition and establish that $\mathcal{W} = \mathfrak{L}(\mathcal{S})$. First note that any member X of $\mathcal{W}$ can be uniquely written as a sum

$$X = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} = U + D + L$$

of a strictly upper block triangular, a block diagonal, and a strictly lower block triangular matrix. Since $\exp(tU) = \begin{bmatrix} I & tB \\ 0 & I \end{bmatrix} \in \Gamma^U \subseteq \mathcal{S}$ for all $t \geq 0$, we conclude that $U \in \mathfrak{L}(\mathcal{S})$, and similarly $L \in \mathfrak{L}(\mathcal{S})$. Clearly $\exp(tD) \in H \subseteq \mathcal{S}$ for all t, so $D \in \mathfrak{L}(\mathcal{S})$ also. Since $\mathfrak{L}(\mathcal{S})$ is a cone, hence closed under addition, we have that $X \in \mathfrak{L}(\mathcal{S})$. Thus $\mathcal{W} \subseteq \mathfrak{L}(\mathcal{S})$.

Conversely suppose that $\exp(tX) \in \mathcal{S}$ for all $t \geq 0$. Using the triple decompositions of Lemma 6.5, we can write

$$\exp(tX) = U(t)D(t)L(t) \text{ for each } t \geq 0.$$

Differentiating both sides with respect to t and evaluating at 0 yields

$$X = \dot{U}(0) + \dot{D}(0) + \dot{L}(0).$$

Then $X_{12} = \dot{U}(0)_{12} = \lim_{t\to 0^+} U(t)_{12}/t \geq 0$, since by equation (1) in the proof of Lemma 6.5 and the following sentences $U(t)$ has its $(1,2)$-entry greater than or equal to 0 for $t \geq 0$. In a similar fashion one argues that $X_{21} \geq 0$. □

Members of $\mathfrak{L}(S)$ are frequently called *Hamiltonian* operators of the sympletic Lie algebra. They are typically the Hamiltonian operators that one considers in the context of continuous systems and differential equations, while the Hamiltonian operators that make up the sympletic semigroup are the ones tht appear in discrete systems. Lie semigroup theory clearly shows the relationship between Hamiltonian operators at the sympletic group level and Hamiltonian operators at the sympletic Lie algebra level.

Suppose that we consider the basic group control equation for a general Lie group modified so that the controls come from some nonempty subset $\Omega \subseteq \mathfrak{g}$, with initial condition $g(0) = e$, the identity of the group. The *attainable set* $A(\Omega)$ is the set of points that appear on trajectories of this system for some $t \geq 0$.

Proposition 8.2. *The attainable set $A = A(\Omega)$ is a subsemigroup of G. If $g(\cdot)$ is a trajectory of the system with $[t_1, t_2]$, $t_1 < t_2$ in its domain, then $g(t_2) = sg(t_1)$ for some $s \in A$.*

Proof. Let $u_i(\cdot) : [0, T_i] \to \Omega$ be steering functions for $i = 1, 2$, and let $g_i(\cdot)$, $i = 1, 2$ be the corresponding trajectories. It is elementary to observe that the concatenation steering function $u = u_1 * u_2 : [0, T_1 + T_2] \to \Omega$ has trajectory given by $g(t) = g_1(t)$ for

$0 \le t \le T_1$ and $g(t) = g_2(t - T_1)g_1(T_1)$ for $T_1 \le t \le T_2$. In particular, $g(T_1 + T_2) = g_2(T_2)g_1(T_1)$, so the attainable set is a semigroup.

Let $u(\cdot)$ be a steering function with domain containing $[t_1, t_2]$ and corresponding trajectory $g(\cdot)$. Define $\gamma(t) = \Phi(t + t_1)(\Phi(t_1))^{-1}$. Define $\tilde{u}(\cdot)$ on $[0, t_2 - t_1]$ by $\tilde{u}(t) = u(t + t_1)$. Then

$$\dot{\gamma}(t) = u(t + t_1)\Phi(t + t_1)(\Phi(t_1))^{-1} = \tilde{u}(t)\gamma(t),\ \gamma(0) = e.$$

It follows that $\gamma(t_2 - t_1) \in A$. But $\gamma(t_2 - t_1) = \Phi(t_2)(\Phi(t_1))^{-1}$, and hence $\Phi(t_2) = s\Phi(t_1)$ for $s = \gamma(t_2 - t_1)$. □

Restricting the preceding argument to the set of steering functions consisting of piecewise constant maps, one observes that the reachable set is the semigroup consisting of finite products of members of $\exp(\Omega)$ (see, for example, [8]). For the case that S is a closed subsemigroup and $\Omega = \mathfrak{L}(S)$, we conclude from the definition of the latter that the attainable set for the class of piecewise constant functions is contained in S. The density of the set of piecewise constant controls and the continuous dependence of solutions on controls then yields that the attainable set is contained in the closed set S. From these observations and Proposition 7.2 we have the first assertion of the following

Proposition 8.3. *Each solution $\Phi(t)$ for $t \ge 0$ of the basic group control equation on* $\mathrm{Sp}(V_E)$

$$\dot{g}(t) = u(t)g(t),\ g(0) = id_{V(E)},\ u(t) \in \mathfrak{L}(\mathcal{S}),$$

is contained in the semigroup $\mathcal{S}$, i.e., the attainable set is contained in $\mathcal{S}$. If $\Phi(s) \in \mathcal{S}_i$ for some s and some $i = 0, 1,$ or 2, then $\Phi(t) \in \mathcal{S}_i$ for all $t > s$.

Proof. Only the last assertion remains to be proved. But this follows from the second assertion of Proposition 8.2 and the fact that each $\mathcal{S}_i$ is an ideal of $\mathcal{S}$. □

Remark 8.4. *If one considers the basic group control equation*

$$\dot{g}(t) = u(t)g(t),\ g(0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix},\ u(t) \in \mathfrak{L}(\mathcal{S}),\ K_0 \in \mathcal{P},$$

then the solution $\Phi(t)g(0)$ evolves in $\mathcal{S}$ for $t \ge 0$ by the preceding proposition and the semigroup property of $\mathcal{S}$. Thus one can form the triple factorization of $\Phi(t)g(0)$ as *given in Lemma 6.5. We note from the proof of that lemma that the* $(1,2)$*-entry of the upper block triangular factor is given by* $g_{12}(t)(g_{22}(t))^{-1}$*, which has initial value* K_0 *and by Lemma 3.2 satisfies the Riccati equation determined by the steering function* $u(\cdot)$*. Thus the function that sends* t *to the* $(1,2)$*-block of the upper triangular factor of the triple decomposition yields the solution of the corresponding Riccati equation. Similar remarks apply over any interval in which* $g_{22}(t)$ *is invertible.*

Our results on the sympletic semigroup lead to a semigroup-theoretic proof of the following global existence result concerning the Riccati equation.

Theorem 8.5. *The Riccati equation*

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t),\ K(t_0) = K_0$$

has a solution in $\mathcal{P}$ *for all* $t \geq t_0$ *if* $R(t), S(t) \geq 0$ *for all* $t \geq t_0$ *and* $K_0 \geq 0$*. If additionally* $K(t_1) \in \mathcal{P}_0$ *for some* $t_1 \geq t_0$*, then* $K(t) \in \mathcal{P}_0$ *for all* $t > t_1$*.*

Proof. The Riccati equation has extended solution on $\mathcal{M}$ given by

$$\tilde{K}(t) = \Phi(t)(\Phi(t_0))^{-1}\begin{pmatrix} K_0 \\ I \end{pmatrix}$$

(see equation (ES) of Section 5). By Proposition 8.2 $\Phi(t)(\Phi(t_0))^{-1} \in \mathcal{S}$ for all $t \geq t_0$. It then follows from Proposition 7.1 that $\tilde{K}(t) \in \mathcal{P}$ for $t \geq t_0$ and thus is equal to $K(t)$ (see Proposition 5.1).

If $K(t_1) \in \mathcal{P}_0$ for some $t_1 \geq t_0$, then for $t > t_1$,

$$\begin{aligned} K(t) &= \tilde{K}(t) = \Phi(t)(\Phi(t_0))^{-1}\begin{pmatrix} K_0 \\ I \end{pmatrix} \\ &= s\Phi(t_1)(\Phi(t_0))^{-1}\begin{pmatrix} K_0 \\ I \end{pmatrix} = sK(t_1), \end{aligned}$$

for some $s \in \mathcal{S}$ by Propositions 8.3 and 8.2. The conclusion follows from Proposition 7.1. □

Remark 8.6. *One typically sees the Riccati equation in a different format: the signs on the right-hand side of the differential equation are all reversed and one desires a solution for* $t \leq t_0$*. This case is easily translated to the one of Theorem 8.5 by time* *reversal about t_0. Our elementary reformulation is more typical and convenient in the semigroup setting. We refer the reader to* [13, Chapter 8] *for a very accessible presentation of a variety of applications of the Riccari equation in control theory, particular the application to optimal control involving regulator (linear quadratic) problems.*

9. The Loewner order

For $X, Y \in \mathrm{Sym}(E)$, we define

$$\begin{aligned} X &< Y :\Longleftrightarrow Y - X \in \mathcal{P}_0, \\ X &\leq Y :\Longleftrightarrow Y - X \in \mathcal{P}. \end{aligned}$$

The order $\leq$ is sometimes called the *Loewner order*. For $X \leq Y$ (respectively, $X < Y$) we define the order intervals

$$\begin{aligned} [X, Y] &= \{Z \in \mathrm{Sym}(E) : X \leq Z \leq Y\}, \\ (X, Y) &= \{Z \in \mathrm{Sym}(E) : X < Z < Y\}, \end{aligned}$$

respectively.

Lemma 9.1. *If $A \in \mathrm{Sym}(E)$ satisfies $\|A\| < 1$, then $I + A \in \mathcal{P}_0$. Hence $\{I + A : \|A\| < 1\}$ is an open set about I in $\mathcal{P}_0$.*

Proof. Pick $r \in \mathbb{R}$ such that $\|A\| < r < 1$. Then by the Cauchy-Schwarz inequality

$$-\langle x, Ax\rangle \leq |\langle x, Ax\rangle| \leq \|x\|\,\|Ax\| \leq \|A\|\,\|x\|^2 \leq r\|x\|^2 = \langle x, rI(x)\rangle,$$

and hence $rI + A \geq 0$. Then also $I + A = (rI + A) + (1 - r)I \geq 0$ and is in $\mathcal{P}_0$ by Lemma 6.1. The last assertion now follows immediately. □

Lemma 9.2. *The set $\mathcal{P}_0$ is open in $\mathrm{Sym}(E)$, and hence open in $\mathcal{M}$, if we identify the symmetric operators with the finite points of $\mathcal{M}$.*

Proof. For $P \in \mathcal{P}_0$ the matrix $\begin{bmatrix} P^{1/2} & 0 \\ 0 & P^{-1/2} \end{bmatrix}$ is in the sympletic semigroup $\mathcal{S}$ and carries I to P and $\mathcal{P}_0$ into $\mathcal{P}_0$ by Proposition 7.1. Thus it carries the open set around I contained in $\mathcal{P}_0$ (Lemma 9.1) onto an open set around P that is contained in $\mathcal{P}_0$. Since $\mathrm{Sym}(E)$ is identified with the open set $\mathcal{M}_0$ of finite points in $\mathcal{M}$ (Proposition 4.4), the last assertion follows. □

Proposition 9.3. *For any* $A, B \in \mathrm{Sym}(E)$ *with* $B < A$,

(i) *the sets* $(-\infty, A) = \{Y \in \mathrm{Sym}(E) : Y < A\}$ *and* $(B, +\infty) = \{Z \in \mathrm{Sym}(E) : B < Z\}$ *are open;*

(ii) *the interval* (B, A) *is open.*

Proof. We have $(-\infty, A) = A - \mathcal{P}_0$ and $(B, \infty) = B + \mathcal{P}_0$, which are open since $\mathcal{P}_0$ is. The intersection of these two sets is (B, A). □

Proposition 9.4. *For any* $A \in \mathcal{P}_0$, *the sets* $\{(-(1/n)A, (1/n)A) : n \in \mathbb{N}\}$ *form a basis of open sets at* 0 *in* $\mathrm{Sym}(E)$.

Proof. The sets $(-(1/n)A, (1/n)A)$ are open by the previous proposition. Suppose that $-(1/n)A < X < (1/n)A$ in $\mathrm{Sym}(E)$. Then there exist $P, Q \in \mathcal{P}_0$ such that $X = P + (-(1/n)A)$ and $X + Q = (1/n)A$. Eliminating X, we obtain $P + Q = (2/n)A$, so $P, Q < (2/n)A$. We have

$$\|P^{1/2}x\|^2 = \langle x, Px\rangle \le \langle x, (2/n)Ax\rangle \le (2/n)\|A\|\,\|x\|^2.$$

It follows that $\|P^{1/2}\| \le (\sqrt{2}/\sqrt{n})\|A\|^{1/2}$, and thus $\|P\| \le \|P^{1/2}\|^2 \le (2/n)\|A\|$. We conclude that

$$\|X\| \le \|P\| + (1/n)\|A\| \le (3/n)\|A\|.$$

Thus the set $(-(1/n)A, (1/n)A)$ is contained in the open ball around 0 of radius $(3/n)\|A\|$. □

Proposition 9.5. *The closure of* $\mathcal{P}_0$ *in* $\mathrm{Sym}(E)$ *is* $\mathcal{P}$.

Proof. That $\mathcal{P}$ is closed in $\mathrm{Sym}(E)$ follows immediately from its definition. Since for $A \in \mathcal{P}$, $A = \lim_{n\to\infty} A + (1/n)I$ and the members of the sequence are in $\mathcal{P}_0$ by Lemma 6.1, the proposition follows. □

A partial order $\le$ on a topological space X is *closed* if $\le\, = \{(x, y) : x \le y\}$ is closed in $X \times X$.

Proposition 9.6. *The Loewner order* $\le$ *is closed on* $\mathrm{Sym}(E)$. *Each order interval* $[A, B] = \{X \in \mathrm{Sym}(E) : A \le X \le B\}$ *for* $A \le B$ *is closed in* $\mathcal{M}$ *(where, as usual, we identify* $\mathrm{Sym}(E)$ *with the finite points* $\mathcal{M}_0$ *of* $\mathcal{M}$*). The interior of* $[A, B]$ *is equal to* (A, B).

Proof. We observe that

$$\{(X,Y) : X \leq Y\} = \{(X,Y) : Y - X \in \mathcal{P}\},$$

and the latter set is closed since $\mathcal{P}$ is (by the previous proposition).

Consider the open set $(-I, I)$ around 0. Since $\mathcal{M}$ is regular (that coset spaces are regular is a standard and elementary result in the theory of topological groups), there exists an open set U containing 0 such that $\overline{U} \subseteq (-I, I)$, where the closure is taken in $\mathcal{M}$. For any $A \in \mathcal{P}$, pick $n > 0$ such that $(-(1/n)A, (1/n)A) \subseteq U$; this is possible by Proposition 9.4. Then $[0, (1/2n)A] = \mathcal{P} \cap ((1/2n)A - \mathcal{P})$ is closed in $\mathrm{Sym}(E)$, contained in $(-(1/n)A, (1/n)A)$ and hence in U, thus closed in $\overline{U}$, and therefore closed in $\mathcal{M}$. The diagonal operator in $\mathrm{Sp}(V_E)$ with entries $\sqrt{2n}I$ and $(1/\sqrt{2n})I$ carries the closed interval $[0, (1/2n)A]$ onto $[0, A]$ and thus the latter is also closed in $\mathcal{M}$. Since any closed interval $[B, A]$ is the image of $[0, A - B]$ under the operator with block matrix entries $\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$, we conclude they are all closed.

Consider a closed interval $[A, B]$ for $A \leq B$. Since (A, B) is open (Proposition 9.3), it is contained in the interior of $[A, B]$. Conversely if X is in the interior of $[A, B]$, then there exists some open set U containing 0 such that $X + U \subseteq [A, B]$. By Proposition 9.4 there exists an $n \in \mathbb{N}$ such that $(-(1/n)I, (1/n)I) \subseteq U$. Then $A \leq X - (1/n)I < X < X + (1/n)I \leq B$, so $A < X < B$. This concludes the proof. □

Proposition 9.7. *For an element $A \in \mathrm{Sym}(E)$, the following are equivalent:*

1. $A \in \mathcal{P}$*;*
2. $A + X$ *is invertible for all* $X \in \mathcal{P}_0$*;*
3. $A + rI$ *is invertible for all* $r > 0$.

Proof. Item (2) follows from item (1) by Lemma 6.1 and item (3) is a trivial consequence of item (2). Assume (3) and suppose that $A \notin \mathcal{P}$. Consider the segment $\{tA + (1-t)I : 0 \leq t \leq 1\}$. This connected segment cannot lie entirely in the union of the two disjoint open sets $\mathcal{P}_0$ and $\mathrm{Sym}(E) \setminus \mathcal{P}$, and thus must meet the set $\mathcal{P} \setminus \mathcal{P}_0$, which by definition consists of noninvertible elements. Hence $tA + (1-t)I$ is not invertible for some $0 < t < 1$. We conclude that the scalar multiple $A + ((1-t)/t)I$ is not invertible, a contradiction. □

Proposition 9.8. *Inversion on $\mathcal{P}_0$ is order reversing.*

Proof. If $A \in \mathcal{P}_0$ and $I \leq A$, then

$$\langle x, A^{-1}x\rangle = \langle A^{-1/2}x, A^{-1/2}x\rangle \leq \langle A^{-1/2}x, A(A^{-1/2}x)\rangle = \langle A^{-1/2}x, A^{1/2}x\rangle = \langle x, x\rangle,$$

and thus $A^{-1} \leq I$. Now

$$X \leq Y \Rightarrow I = X^{-1/2}XX^{-1/2} \leq X^{-1/2}YX^{-1/2}.$$

Thus inverting, we see that

$$I \geq X^{1/2}Y^{-1}X^{1/2} \Rightarrow X^{-1} = X^{-1/2}IX^{-1/2} \geq Y^{-1}.$$

Hence inversion on $\mathcal{P}_0$ is order reversing. □

Proposition 9.9. *The closure $\overline{\mathcal{P}}$ of $\mathcal{P}$ in $\mathcal{M}$ has interior $\mathcal{P}_0$.*

Proof. Since $\mathcal{P}_0$ is open in $\mathcal{M}$ (Lemma 9.2), it is contained in the interior of $\overline{\mathcal{P}}$. For the converse, we consider the sympletic maps on $\mathcal{M}$ given by

$$t_I := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}, \; t_{-I} := \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}, \; J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Note that

$$Jt_I \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} D \\ -(B+I) \end{pmatrix}; \; Jt_I \begin{pmatrix} X \\ I \end{pmatrix} = \begin{pmatrix} I \\ -(X+I) \end{pmatrix} = \begin{pmatrix} -(I+X)^{-1} \\ I \end{pmatrix}, \; X \in \mathcal{P}.$$

Since inversion is order reversing on $\mathcal{P}_0$ and $X \to -X$ is order reversing on $\mathrm{Sym}(E)$, we conclude that Jt_I is order preserving on $\mathcal{P}$.

We observe that

$$Jt_I \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} -I \\ I \end{pmatrix}; \; Jt_I \begin{pmatrix} (n-1)I \\ I \end{pmatrix} = \begin{pmatrix} (-1/n)I \\ I \end{pmatrix}.$$

We conclude that

$$Jt_I(\mathcal{P}) \subseteq \bigcup_n \{X \in \mathrm{Sym}(E) : -I < X < \frac{-1}{n} I\}$$

since $\mathcal{P} = \bigcup [0, nI)$. (If $A \in \mathcal{P}_0$, then $(1/n)A \in (-I, I)$ for some n by Proposition 9.4, and so $A < nI$.) Thus $\overline{Jt_I(\mathcal{P})} \subseteq \overline{[-I, 0)} \subseteq [-I, 0]$.

Suppose that $\begin{pmatrix} B \\ D \end{pmatrix}$ is in the interior of $\overline{\mathcal{P}}$, the closure taken in $\mathcal{M}$. Then

$$Jt_I \begin{pmatrix} B \\ D \end{pmatrix} \in \operatorname{int}\overline{Jt_I(\mathcal{P})} \subseteq \operatorname{int}[-I,0] = (-I,0),$$

the last equality coming from Proposition 9.6. Hence $Jt_I \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} -P \\ I \end{pmatrix}$ for some $P \in \mathcal{P}$, $0 < P < I$. Since the inverse of Jt_I is given by $t_{-I}(-J)$, we have

$$\begin{pmatrix} B \\ D \end{pmatrix} = t_{-I}(-J) \begin{pmatrix} -P \\ I \end{pmatrix} = t_{-I} \begin{pmatrix} -I \\ -P \end{pmatrix} = t_{-I} \begin{pmatrix} P^{-1} \\ I \end{pmatrix} = \begin{pmatrix} P^{-1} - I \\ I \end{pmatrix}.$$

Since $P < I$ implies $I < P^{-1}$, we conclude that $P^{-1} - I > 0$, and thus $\begin{pmatrix} B \\ D \end{pmatrix} \in \mathcal{P}_0$. $\square$

The next proposition gives another important property of the symplectic semigroup.

Proposition 9.10. *Members of the symplectic semigroup $\mathcal{S}$ satisfy the following mono tonicity properties:*

(i) *For $g \in \mathcal{S}$ and $X, Y \in \mathcal{P}_0$, $X \leq Y$ if and only if $g(X) \leq g(Y)$.*
(ii) *For $g \in \mathcal{S}$ and $X, Y \in \mathcal{P}$, $X \leq Y$ implies $g(X) \leq g(Y)$.*

Proof. (i) We verify this for each of the factors in the triple decomposition of Lemma 6.5. This is straightforward for the upper triangular and diagonal factors. Suppose $X, Y \in \mathcal{P}_0$ and $X \leq Y$. Then for $C \in \mathcal{P}$,

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{pmatrix} X \\ I \end{pmatrix} = \begin{pmatrix} X \\ CX + I \end{pmatrix} = \begin{pmatrix} X(CX+I)^{-1} \\ I \end{pmatrix},$$

and similarly the image of Y is $Y(CY+I)^{-1} = (C + Y^{-1})^{-1}$. Since by the previous proposition, inversion is order reversing on $\mathcal{P}_0$, we conclude that $C + X^{-1} \geq C + Y^{-1}$ and therefore $(C + X^{-1})^{-1} \leq (C + Y^{-1})^{-1}$. These steps are reversible. Hence lower triangular matrices in $\mathcal{S}$ also preserve the order on $\mathcal{P}_0$.

(ii) For $X \leq Y$ in $\mathcal{P}$, we have $X + (1/n)I \leq Y + (1/n)I$ for each $n > 0$. By the previous paragraph and Proposition 9.7, $g(X + (1/n)I) \leq g(X + (1/n)Y$ for each n. Since the order relation $\leq$ is closed (Proposition 9.6), we have by taking the limit as $n \to \infty$ that $g(X) \leq g(Y)$. $\square$

10. Order and the Riccati equation

In this section we briefly consider relationships that exist between the Riccati equation and the Loewner order.

If the Riccati equation

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), K(t_0) = K_0$$

has a solution on the interval $[t_0, t_1]$, then we denote $K(t_1)$ by $\Gamma(t_0, K_0, t_1)$.

Proposition 10.1. *Suppose in the Riccati equation that $R(t), S(t) \geq 0$ for all $t \in \mathbb{R}$ and suppose that $K_0 \geq 0$. Then for all $t_0 \leq t_1$, $\Gamma(t_0, K_0, t_1)$ exists and is in $\mathcal{P}$. Futhermore, for $K_0 = 0$ we have*

$$\Gamma(t_1, 0, t_2) \leq \Gamma(t_0, 0, t_2) \textit{ for all } t_0 < t_1 < t_2.$$

Thus the mapping $t \mapsto \Gamma(t, 0, t_1) : (-\infty, t_1] \to \mathcal{P}$ is a continuous order-reversing map.

Proof. The first assertion follows from Theorem 8.5. By Proposition 5.2 we have that $\Gamma(t_0, 0, t_2) = \Phi(t_2)(\Phi(t_0))^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}$ and $\Gamma(t_1, 0, t_2) = \Phi(t_2)(\Phi(t_1))^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}$. By Proposition 8.2 there exist $s, s' \in \mathcal{S}$ such that $\Phi(t_2) = s\Phi(t_1)$ and $\Phi(t_1) = s'\Phi(t_0)$. Then

$$\begin{aligned} \Gamma(t_1, 0, t_2) &= \Phi(t_2)(\Phi(t_1))^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} = s \begin{pmatrix} 0 \\ I \end{pmatrix} \\ &\leq ss' \begin{pmatrix} 0 \\ I \end{pmatrix} = \Phi(t_2)(\Phi(t_0))^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} = \Gamma(t_0, 0, t_2), \end{aligned}$$

where the inequality follows from the facts that s is order-preserving on $\mathcal{P}$ (Proposition 9.10), $s'(0) \in \mathcal{P}$ (Proposition 7.1) and 0 is the least element in $\mathcal{P}$. Since $\Gamma(t, 0, t_1) = \Phi(t_1)(\Phi(t))^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}$, we conclude that the map $t \mapsto \Gamma(t, 0, t_1)$ is continuous on $(-\infty, t_1)$. □

We recall another important connection of the Loewner order with the Riccati equation. The elegant, quick proof is taken from [2], although the theorem appeared earlier in [10].

Proposition 10.2. *Consider the Hamiltonian matrices in the symmetric Lie algebra* $\mathfrak{sp}(V_E)$

$$H(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A^*(t) \end{bmatrix}, \text{ and } \tilde{H}(t) = \begin{bmatrix} \tilde{A}(t) & \tilde{R}(t) \\ \tilde{S}(t) & -\tilde{A}^*(t) \end{bmatrix}$$

and the corresponding Riccati equations

$$\begin{aligned} \dot{K}(t) &= R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), K(0) = K_0, \\ \dot{\tilde{K}}(t) &= \tilde{R}(t) + \tilde{A}(t)\tilde{K}(t) + \tilde{K}(t)\tilde{A}^*(t) - \tilde{K}(t)\tilde{S}(t)\tilde{K}(t), \tilde{K}(0) = \tilde{K}_0. \end{aligned}$$

Assume that

$$\tilde{H}J \leq HJ, \text{ i.e., } \begin{bmatrix} \tilde{R} - R & A - \tilde{A} \\ A^* - \tilde{A}^* & S - \tilde{S} \end{bmatrix} \geq 0$$

and $0 \leq K_0 \leq \tilde{K}_0$. *Then for every* $t \geq 0$, *we have* $K(t) \leq \tilde{K}(t)$.

Proof. Global solutions $K(\cdot)$ and $\tilde{K}(\cdot)$ exist for all $t \geq 0$ by Theorem 8.5. The symmetric operator function $U(t) := \tilde{K}(t) - K(t)$ satisfies the Riccati differential equation

$$\dot{U} = (\tilde{A} - K\tilde{S})U + U(\tilde{A} - K\tilde{S})^* - U\tilde{B}U + \begin{bmatrix} I & -X \end{bmatrix} (HJ - \tilde{H}J) \begin{bmatrix} I \\ -X \end{bmatrix}$$

with a positive semidefinite intial condition. The result now follows from Theorem 8.5. □

References

[1] P. Bougerol, "Kalman filtering with random coefficients and contractions, *SIAM J. Control and Optimization* **31** (1993), 942–959.

[2] L. Dieci and T. Eirola, "Preserving monotonicity in the numerical solution of Riccati differential equations," Numer. Math. **74** (1996), 35–47.

[3] R. Hermann, *Cartanian Geometry, Nonlinear Waves, and Control Theory*, Part A, Interdisciplinary Mathematics **20**, Math. Sci. Press, Boorline, MA, 1979.

[4] J. Hilgert, K. H. Hofmann, and J. D. Lawson, *Lie Groups, Convex Cones, and Semigroups*, Oxford Press, Oxford, 1989.

[5] J. Hilgert and K.-H. Neeb, *Basic Theory of Lie Semigroups and Applications*, Springer Lecture Notes in Mathematics **1552**, Springer, 1993.

[6] V. Jurdjevic, *Geometric Control Theory*, Cambridge Press, Cambridge, 1997.

[7] K. Koufany, "Semi-groupe de Lie associe a un cone symmetrique," *Ann. Inst. Fourier*, Grenoble **45** (1995), 1–29.

[8] J. Lawson, "Geometric control and Lie semigroup theory," In: *Differential Geometry and Control, Proceedings of Symposia in Pure Mathematics*, Volume 64, Editors: G. Ferreyra, R. Gardner, H. Hermes, H. Sussmann, Amer. Math. Soc., 1999, 207–221.

[9] J. Lawson and Y. Lim, "Lie semigroups with triple decomposition," *Pacific J. Math.* **194** (2000), 393–412.

[10] W. T. Reid, "Monotoneity properties of solutions of Hermitian Riccati equations," SIAM J. Math. Analysis **1** (1970), 195–213.

[11] Y. Sachkov, "Survey on controllability of invariant systems on solvable Lie groups," In: *Differential Geometry and Control, Proceedings of Symposia in Pure Mathematics*, Volume 64, Editors: G. Ferreyra, R. Gardner, H. Hermes, H. Sussmann, Amer. Math. Soc., 1999, 297–318.

[12] M. Shayman, "Phase portrait of the matrix Riccati equation," *SIAM J. Control Optim.* **24** (1986), 1–65.

[13] E. D. Sontag, *Mathematical Control Theory*, 2nd Edition, Springer, Berlin, 1998.

[14] M. Wojtkowski, "Invariant families of cones and Lyapunov exponents," *Ergod. Th. and Dynam. Sys.* **5** (1985), 145–161.

[15] —, "Measure theoretic entropy of the system of hard spheres," *Ergod. Th. and Dynam. Sys.* **8** (1988), 133–153.