Name:____

Instructions. *Show all work in the space provided: credit is given only for what you write on your paper.* Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading. No books or notes are allowed. A scientific calculator is ok - but not needed . The *maximum total score* is 100.

1. (20 points) Find the limit, if it exists, *showing all work needed to justify your conclusion*, or else show the limit does not exist *by finding two different limits along suitable paths.*

   (a) $\lim_{(x,y)\to(0,0)} \frac{3xy}{\sqrt{x^2+y^2}}$

   (b) $\lim_{(x,y)\to(0,0)} \frac{x^3y+y^2x^2}{x^4+y^4}$

2. (20 points) For both parts (a) and (b) below, let

$$F(x,y,z) = \arctan(xz) - 2y + z.$$

   (a) Find $\frac{\partial F}{\partial z}$.

   (b) If $z = f(x,y)$ is a differentiable solution of the equation

$$F(x,y,z) = \frac{\pi}{4} - 1$$

   find $\frac{\partial z}{\partial y}$.

3. (20 points) Let $g(u, v) = f(x, y)$ where f is differentiable,

$$x = u + \ln v \text{ and } y = v + e^u.$$

Use the table of values to find $g_u(0, 1)$ *and* $g_v(0, 1)$ (Hint: Write the appropriate form of the Chain Rule and evaluate at the points corresponding to $u = 0$ and $v = 1$. You may think of $z = f(x, y)$ and $g_u(0, 1) = \frac{\partial z}{\partial u}(0, 1)$, etc.)

point \ function	f	g	f_x	f_y
(0,1)	1	2	3	4
(0,2)	-1	-2	-3	-4

4. (20 points) Let $F(x, y, z) = 2x^2 + 3y^2 + z^2$ for both parts below.

(a) Find $\vec{\nabla} F(x, y, z)$.

(b) Find an equation for the plane tangent to the graph of

$$F(x, y, z) = 23$$

at the point $(1, 2, 3)$.

5. (20 points) Find the *maximum* value and the *minimum* value of the the function

$$f(x, y) = xy\left(9 - x^2 - y^2\right)$$

defined on the domain D in the xy-plane which is determined by the inequality $x^2 + y^2 \leq 9$.

Solutions

1. (20 points) Find the limit, if it exists, *showing all work needed to justify your conclusion*, or else show the limit does not exist *by finding two different limits along suitable paths.*

   (a) $\lim_{(x,y)\to(0,0)} \frac{3xy}{\sqrt{x^2+y^2}}$

   Solution: We convert the problem to polar coordinates as follows. $\lim_{(x,y)\to(0,0)} \frac{3xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{3r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} 3r\cos\theta\sin\theta = 0$ since $r \longrightarrow 0$ and the trigonometric factors are bounded.

   (b) $\lim_{(x,y)\to(0,0)} \frac{x^3y+y^2x^2}{x^4+y^4}$

   Solution: We show as follows that this limit does not exist. Along the path $(x, y) = (x, 0) \longrightarrow (0, 0)$ we obtain the limit 0. However, along the path $(x, y) = (x, x) \longrightarrow (0, 0)$ we obtain the limit $1 \neq 0$.

2. (20 points) For both parts (a) and (b) below, let

$$F(x, y, z) = \arctan(xz) - 2y + z.$$

   (a) Find $\frac{\partial F}{\partial z}$.

   Solution: $\frac{\partial F}{\partial z} = \frac{x}{1+(xz)^2} + 1 = \frac{1+x+x^2z^2}{1+x^2z^2}$.

   (b) If $z = f(x, y)$ is a differentiable solution of the equation

$$F(x, y, z) = \frac{\pi}{4} - 1$$

   find $\frac{\partial z}{\partial y}$.

   Solution: $F_y + F_z z_y = 0$, so $\frac{\partial z}{\partial y} = z_y = -\frac{F_y}{F_z} = \frac{2\left(1+x^2z^2\right)}{1+x+x^2z^2}$.

3. (20 points) Let $g(u,v) = f(x,y)$ where f is differentiable,

$$x = u + \ln v \text{ and } y = v + e^u.$$

Use the table of values to find $g_u(0,1)$ *and* $g_v(0,1)$ (Hint: Write the appropriate form of the Chain Rule and evaluate at the points corresponding to $u = 0$ and $v = 1$. You may think of $z = f(x,y)$ and $g_u(0,1) = \frac{\partial z}{\partial u}(0,1)$, etc.)

point \ function	f	g	f_x	f_y
(0,1)	1	2	3	4
(0,2)	-1	-2	-3	-4

Solution:

$$g_u(0,1) = f_x(0,2)x_u(0,1) + f_y(0,2)y_u(0,1) = (-3)1 + (-4)1 = -7$$

and

$$g_v(0,1) = f_x(0,2)x_v(0,1) + f_y(0,2)y_v(0,1) = (-3)1 + (-4)1 = -7.$$

4. (20 points) Let $F(x,y,z) = 2x^2 + 3y^2 + z^2$ for both parts below.

   (a) Find $\vec{\nabla} F(x,y,z)$.
   Solution: $\vec{\nabla} F(x,y,z) = (4x, 6y, 2z) = 4x\vec{i} + 6y\vec{j} + 2z\vec{k}$.

   (b) Find an equation for the plane tangent to the graph of

   $$F(x,y,z) = 23$$

   at the point $(1,2,3)$.
   Solution: A normal to the plane is given by

   $$\vec{\nabla} F(1,2,3) = (4,12,6).$$

   Hence an equation of the tangent plane is given by

   $$4(x-1) + 12(y-2) + 6(z-3) = 0,$$

   or

   $$2x + 6y + 3z = 23.$$

5. (20 points) Find the *maximum* value and the *minimum* value of the the function

$$f(x, y) = xy\left(9 - x^2 - y^2\right)$$

defined on the domain D in the xy-plane which is determined by the inequality $x^2 + y^2 \leq 9$.

Solution: Observe that the domain D is a closed and bounded circular disk and f is continuous on D. Also, we have $f(x, y) = xy\,(9 - x^2 - y^2)$, which is identically zero on the boundary curve of D and also on the two axes. Since $f(x, y)$ does have strictly positive values and strictly negative values, the absolute maximum and minimum values of f must occur at *interior* points for which $x \neq 0$ and $y \neq 0$. Since f has both partial derivatives everywhere, we can find the extreme points among the solutions of the two simultaneous equations

$$f_x = y\left(9 - 3x^2 - y^2\right) = 0$$

$$f_y = x\left(9 - x^2 - 3y^2\right) = 0.$$

Since $x \neq 0$ and $y \neq 0$ at an extreme point, we solve the pair of equations

$$3x^2 + y^2 = 9$$

$$x^2 + 3y^2 = 9$$

The *commonest error* was to be unable to solve this system. Probably the easiest way is to subtract the second equation from 3 times first, yielding $x^2 = \frac{18}{8} = \frac{9}{4}$, or $x = \pm\frac{3}{2}$ and $y = \pm\frac{3}{2}$. Thus the maximum value of the function is given by

$$f\left(\frac{3}{2}, \frac{3}{2}\right) = f\left(-\frac{3}{2}, -\frac{3}{2}\right) = \frac{81}{8}.$$

And the minimum value of f is given by

$$f\left(\frac{3}{2}, -\frac{3}{2}\right) = f\left(-\frac{3}{2}, \frac{3}{2}\right) = -\frac{81}{8}.$$

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8				
80-89 (B)	6				
70-79 (C)	14				
60-69 (D)	9				
0-59 (F)	2				
Test Avg	76.9%	%	%	%	%
Quiz Avg	75.4%				