Instructions. Show all work in the space provided: credit is given only for what you write on your paper. Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading. No books or notes are allowed. A scientific calculator is ok - but not needed. There are 5 (five) problems: maximum total score = 100.

1. (20 points) Let \( f(x, y) = x^3y + 12x^2 - 8y. \)

   (a) Find all the first order and second order partial derivatives of \( f. \)

   (b) Find all critical points and use the second derivative test to classify each one as either a saddle point, a local maximum point, or a local minimum point of \( f. \)
2. (20 points) Find
\[ \int\int_D \frac{3y^2}{x^4 + 1} \, dA \]
where \( D = \{(x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq x\} \).

3. (20 points) Use polar coordinates to find
\[ \int\int_D \sin (x^2 + y^2) \, dA \]
where \( D \) is the region to the right of the \( y \)-axis but inside the graph of \( x^2 + y^2 = 16 \).
4. (20 points) Find $\iiint_{R} z \, dV$ if $R$ is the region bounded by the three planes $z = 0$, $z = y$, and $y = 1$, and the parabolic cylinder $y = x^2$. 
5. (20 points) Let $D$ be the square region in the $xy$-plane with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$.

(a) (6) Find and either sketch or describe the region $R$ in the $uv$-plane which corresponds to $D$ if

$$u = x + y \text{ and } v = x - y$$

(b) (6) Find $\frac{\partial (x,y)}{\partial (u,v)}$ using the definitions of $u$ and $v$ in part (a).

(c) (8) Use (a) and (b) to express $\iint_D \cos(x + y)\, dx\, dy$ as a double integral over $R$ with respect to $u$ and $v$ and evaluate.

Solutions
1. (20 points) Let \( f(x, y) = x^3y + 12x^2 - 8y \).

(a) Find all the first order and second order partial derivatives of \( f \).

Solution:
\[
\begin{align*}
f_x &= 3x^2y + 24x \\
f_y &= x^3 - 8 \\
f_{xx} &= 6xy + 24 \\
f_{xy} &= f_{yx} = 3x^2 \\
f_{yy} &= 0
\end{align*}
\]

(b) Find all critical points and use the second derivative test to classify each one as either a saddle point, a local maximum point, or a local minimum point of \( f \).

Solution: We need to solve simultaneously the system
\[
\begin{align*}
3x(xy + 8) &= 0 \\
x^3 - 8 &= 0
\end{align*}
\]
The second equation implies that \( x = 2 \) and then the first equation tells us that \( y = -4 \). Thus the only critical point is \((2, -4)\). The discriminant
\[
D = \det \begin{pmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix} = -9x^4 < 0
\]
at the only critical point, which is therefore a saddle point. (See Figure 1.)

2. (20 points) Find
\[
\int \int_D \frac{3y^2}{x^4 + 1} \, dA
\]
where \( D = \{(x, y) | 0 \leq x \leq 1, \, 0 \leq y \leq x\} \).

Solution: \( \int \int_D \frac{3y^2}{x^4 + 1} \, dA = \int_0^1 \int_0^x \frac{3y^2}{x^4 + 1} \, dy \, dx = \int_0^1 \frac{x^3}{x^4 + 1} \, dx = \frac{1}{4} \int_1^2 \frac{du}{u} = \frac{1}{4} \log 2. \)
3. (20 points) Use polar coordinates to find

\[ \iint_{D} \sin \left( x^2 + y^2 \right) \, dA \]

where \( D \) is the region to the right of the \( y \)-axis but inside the graph of \( x^2 + y^2 = 16 \).

Solution:

\[
\iint_{D} \sin \left( x^2 + y^2 \right) \, dA = \int_{-\pi/2}^{\pi/2} \int_{0}^{4} \sin (r^2) \, rdr \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} \left[ \frac{1}{2} (1 - \cos 16) \right] d\theta = \frac{\pi}{2} (1 - \cos 16).
\]
4. (20 points) Find \( \iiint_{R} z \, dV \) if \( R \) is the region bounded by the three planes \( z = 0 \), \( z = y \), and \( y = 1 \), and the parabolic cylinder \( y = x^2 \).

Solution:
\[
\iiint_{R} z \, dV = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{y} z \, dz \, dy \, dx = \int_{-1}^{1} \int_{x^2}^{1} \frac{y^2}{2} \, dy \, dx = \frac{1}{6} \int_{-1}^{1} 1 - x^6 \, dx
\]
\[
= \frac{1}{6} \left( x - \frac{x^7}{7} \right) \bigg|_{-1}^{1} = \frac{2}{7}.
\]

5. (20 points) Let \( D \) be the square region in the \( xy \)-plane with vertices at \((\pm 1, 0)\) and \((0, \pm 1)\).

(a) (6) Find and either sketch or describe the region \( R \) in the \( uv \)-plane which corresponds to \( D \) if

\[
u = x + y \quad \text{and} \quad v = x - y
\]

Solution: \( R \) is the square region bounded by the lines \( u = \pm 1 \) and \( v = \pm 1 \).

(b) (6) Find \( \frac{\partial (x,y)}{\partial (u,v)} \) using the definitions of \( u \) and \( v \) in part (a).

Solution: We calculate that \( x = \frac{u+v}{2} \) and \( y = \frac{u-v}{2} \), so that \( \frac{\partial (x,y)}{\partial (u,v)} = \det \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right) = -\frac{1}{2} \).

(c) (8) Use (a) and (b) to express \( \iint_{D} \cos(x+y) \, dx \, dy \) as a double integral over \( R \) with respect to \( u \) and \( v \) and evaluate.

Solution:
\[
\iint_{D} \cos(x+y) \, dx \, dy = \iint_{R} \cos u \frac{1}{2} \, du \, dv
\]
\[
= \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \cos u \, du \, dv = 2 \sin 1.
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