Name:____

Instructions. *Show all work in the space provided: credit is given only for what you write on your paper.* Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading. No books or notes are allowed. A scientific calculator is ok - but not needed . There are **5 (five)** problems: *maximum total score* = 100.

1. (20 points) Let $f(x, y) = x^3y + 12x^2 - 8y$.

   (a) Find *all* the first order *and* second order partial derivatives of f.

   (b) Find *all critical points* and *use the second derivative test to classify* each one as either a *saddle point*, a *local maximum point*, or a *local minimum point of f*.

2. (20 points) Find

$$\iint_D \frac{3y^2}{x^4+1}\, dA$$

where $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$.

3. (20 points) Use polar coordinates to find

$$\iint_D \sin\left(x^2+y^2\right)\, dA$$

where D is the region to the *right* of the y-axis but *inside* the graph of

$$x^2+y^2=16.$$

4. (20 points) Find $\iiint_R z\, dV$ if R is the region bounded by the three planes $z = 0, z = y,$ and $y = 1$, and the parabolic cylinder $y = x^2$.

5. (20 points) Let D be the square region in the xy-plane with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$.

   (a) (6) Find and either sketch or describe the region R in the uv-plane which corresponds to D if

   $$u = x + y \text{ and } v = x - y$$

   (b) (6) Find $\frac{\partial(x,y)}{\partial(u,v)}$ using the definitions of u and v in part (a).

   (c) (8)Use (a) and (b) to express $\iint_D \cos(x+y)\, dx\, dy$ as a double integral over R with respect to u and v and evaluate.

Solutions

1. (20 points) Let $f(x, y) = x^3y + 12x^2 - 8y$.

   (a) Find *all* the first order *and* second order partial derivatives of f.
   Solution:

$$f_x = 3x^2y + 24x$$
$$f_y = x^3 - 8$$
$$f_{xx} = 6xy + 24$$
$$f_{xy} = f_{yx} = 3x^2$$
$$f_{yy} = 0$$

   (b) Find *all critical points* and *use the second derivative test to classify* each one as either a *saddle point*, a *local maximum point*, or a *local minimum point of f*.
   Solution: We need to solve simultaneously the system

$$\begin{aligned} 3x(xy + 8) &= 0 \\ x^3 - 8 &= 0 \end{aligned}$$

   The second equation implies that $x = 2$ and then the first equation tells us that $y = -4$. Thus the only critical point is $(2, -4)$. The discriminant

$$D = \det \begin{pmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix} = -9x^4 < 0$$

   at the only critical point, which is therefore a *saddle point*. (See Figure 1.)

2. (20 points) Find

$$\iint_D \frac{3y^2}{x^4 + 1} \, dA$$

   where $D = \{(x, y) \mid 0 \le x \le 1, \ 0 \le y \le x\}$.
   Solution: $\iint_D \frac{3y^2}{x^4+1} \, dA = \int_0^1 \int_0^x \frac{3y^2}{x^4+1} \, dy \, dx = \int_0^1 \frac{x^3}{x^4+1} dx = \frac{1}{4} \int_1^2 \frac{du}{u} = \frac{1}{4} \log 2$.

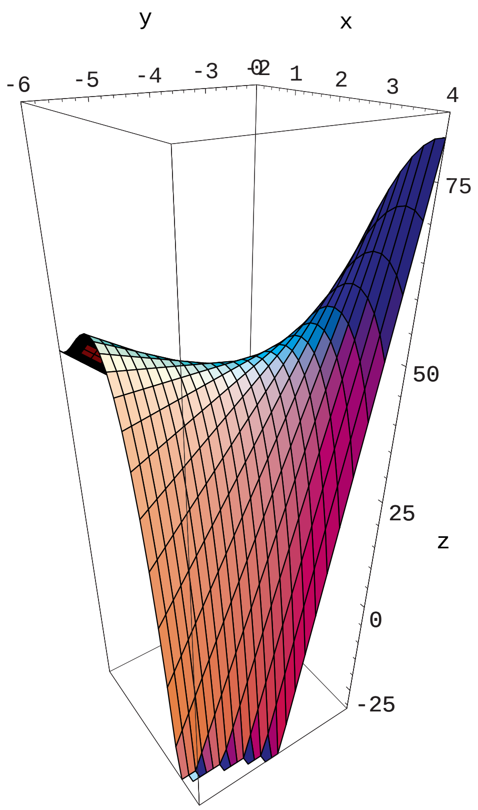

Figure 1: $f(x, y) = x^3y + 12x^2 - 8y$

3. (20 points) Use polar coordinates to find

$$\iint_D \sin\left(x^2 + y^2\right) \, dA$$

where D is the region to the *right* of the y-axis but *inside* the graph of

$$x^2 + y^2 = 16.$$

Solution:

$$\iint_D \sin\left(x^2 + y^2\right) \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 \sin\left(r^2\right) r dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{16} \frac{\sin u}{2} \, du \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos 16}{2} \, d\theta = \frac{\pi}{2}(1 - \cos 16).$$

4. (20 points) Find $\iiint_R z\,dV$ if R is the region bounded by the three planes $z=0, z=y,$ and $y=1$, and the parabolic cylinder $y=x^2$.

Solution:

$$\iiint_R z\,dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{y} z\,dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1}\frac{y^2}{2}\,dy\,dx = \frac{1}{6}\int_{-1}^{1} 1-x^6\,dx$$

$$= \frac{1}{6}\left(x-\frac{x^7}{7}\right)\Bigg|_{-1}^{1} = \frac{2}{7}.$$

5. (20 points) Let D be the square region in the xy-plane with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$.

   (a) (6) Find and either sketch or describe the region R in the uv-plane which corresponds to D if

   $$u = x+y \text{ and } v = x-y$$

   Solution: R is the square region bounded by the lines $u=\pm 1$ and $v=\pm 1$.

   (b) (6) Find $\frac{\partial(x,y)}{\partial(u,v)}$ using the definitions of u and v in part (a).

   Solution: We calculate that $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$, so that $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix} = -\frac{1}{2}$.

   (c) (8)Use (a) and (b) to express $\iint_D \cos(x+y)\,dx\,dy$ as a double integral over R with respect to u and v and evaluate.

   Solution:

   $$\iint_D \cos(x+y)\,dx\,dy = \iint_R \cos u\,\frac{1}{2}\,du\,dv$$

   $$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\cos u\,du\,dv = 2\sin 1.$$

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	7			
80-89 (B)	6	10			
70-79 (C)	14	13			
60-69 (D)	9	7			
0-59 (F)	2	0			
Cumulative Test Avg	76.9%	78.3%	%	%	%
Cumulative Quiz Avg	72.2%	78.0%			