Name:____

Instructions. *Show all work in the space provided: credit is given only for what you write on your paper.* Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading. No books or notes are allowed. A scientific calculator is ok - but not needed . There are **5 (five)** problems: *maximum total score* = 100.

1. (20 points) Find $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ if $\vec{F}(x, y, z) = y\vec{i} + x\vec{j} + (x + y)\vec{k}$ and $\mathcal{C}$ is the line segment from $(1, 0, 2)$ to $(4, 4, 7)$. (Hint: Parameterize the line segment.)

2. (20 points) Let $\vec{F}(x, y) = 3x^2y\vec{\imath} + (x^3 + y^2)\,\vec{\jmath}$.

   (a) (5) Show a simple calculation which confirms that $\int_{(0,0)}^{(2,3)} \vec{F} \cdot d\vec{r}$ is path-independent.

   (b) (10) Find a *potential function* f for $\vec{F}$.

   (c) (5) Evaluate $\int_{(0,0)}^{(2,3)} \vec{F} \cdot d\vec{r}$.

3. (20 points) Use Green's theorem to evaluate

$$\oint_{\mathcal{C}} \left(e^{-x^2} - \frac{y^2}{2}\right)\, dx + (xy + \arctan\sqrt{y})\, dy$$

   where $\mathcal{C}$ is the positively oriented boundary of the triangular region with the vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$.

4. (20 points) Let $\mathcal{S}$ denote the part of the surface $z = 1 - x^2 - y^2$ which is *above* the xy-plane. Let

$$\vec{F}(x, y, z) = y\vec{\imath} - x\vec{\jmath} + (x^2 + y^2)\vec{k}.$$

Find the *flux* $\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$ of $\vec{F}$ across the surface $\mathcal{S}$, which is *oriented upward* by the choice of $\vec{n}$. (Hint: Find $\vec{n}$ and $\vec{F} \cdot \vec{n}$. Then integrate over the surface.)

5. (20 points) Let

$$\vec{F}(x,y,z) = (e^x - y)\,\vec{i} + (e^y - z)\,\vec{j} + (e^z - x)\,\vec{k}$$

Let $\mathcal{C}$ be the positively oriented triangle which bounds the first octant portion of the plane $\mathcal{S}$ given by

$$x + 2y + z = 2$$

with the normal $\vec{n}$ pointing away from the origin.

(a) (5) Find the *curl* of $\vec{F}$.

(b) (5) Find the vector $\vec{n}$.

(c) (10) Use Stokes' theorem to find $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ by evaluating a surface integral over $\mathcal{S}$.

Solutions

1. (20 points) Find $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = y\vec{i} + x\vec{j} + (x+y)\vec{k}$ and $\mathcal{C}$ is the line segment from $(1,0,2)$ to $(4,4,7)$. (Hint: Parameterize the line segment.)

Solution: This integral is *not* path-independent: For example

$$M_z = 0 \neq 1 = P_x$$

Observe that the line can be parameterized as

$$x = 1 + 3t,\ \ y = 4t,\ \ z = 2 + 5t,\ \ 0 \leq t \leq 1$$

Thus

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_0^1 12t+4(1+3t)+5(1+7t)\,dt = 9t + \left.\frac{59t^2}{2}\right|_0^1 = 9+\frac{59}{2} = \frac{77}{2}$$

2. (20 points) Let $\vec{F}(x,y) = 3x^2y\vec{\imath} + (x^3+y^2)\,\vec{\jmath}$.

   (a) (5) Show a simple calculation which confirms that $\int_{(0,0)}^{(2,3)} \vec{F}\cdot d\vec{r}$ is path-independent.

   Solution: $M_y = 3x^2 = N_x$.

   (b) (10) Find a *potential function* f for $\vec{F}$.

   Solution: Since $f_y = 3x^2y$, $f(x,y) = x^3y + h(y)$. Thus

   $$f_y = x^3 + h'(y) = x^3 + y^2$$

   so that $f(x,y) = x^3y + \frac{y^3}{3}$. This is easy to check.

   (c) (5) Evaluate $\int_{(0,0)}^{(2,3)} \vec{F}\cdot d\vec{r}$.

   Solution: $\int_{(0,0)}^{(2,3)} \vec{F}\cdot d\vec{r} = f(2,3) - f(0,0) = 33$.

3. (20 points) Use Green's theorem to evaluate

$$\oint_{\mathcal{C}} \left(e^{-x^2} - \frac{y^2}{2}\right)\,dx + \left(xy + \arctan\sqrt{y}\right)\,dy$$

where $\mathcal{C}$ is the positively oriented boundary of the triangular region with the vertices $(0,0)$, $(1,0)$ and $(1,2)$.

Solution: By Green's theorem,

$$\oint_{\mathcal{C}} \left(e^{-x^2} - \frac{y^2}{2}\right)\,dx + \left(xy + \arctan\sqrt{y}\right)\,dy = \int_0^1 \int_0^{2x} 2y\,dy\,dx$$

$$= \int_0^1 4x^2\,dx = \frac{4}{3}.$$

4. (20 points) Let $\mathcal{S}$ denote the part of the surface $z = 1 - x^2 - y^2$ which is *above* the xy-plane. Let

$$\vec{F}(x, y, z) = y\vec{i} - x\vec{j} + (x^2 + y^2)\vec{k}.$$

Find the *flux* $\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$ of $\vec{F}$ across the surface $\mathcal{S}$, which is *oriented upward* by the choice of $\vec{n}$.(Hint: Find $\vec{n}$ and $\vec{F} \cdot \vec{n}$. Then integrate over the surface.)

Solution: The upward unit normal is given by

$$\vec{n} = \frac{2x\vec{i} + 2y\vec{j} + \vec{k}}{\sqrt{1 + 4\,(x^2 + y^2)}}.$$

Thus

$$\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = \iint_{\mathcal{S}} \frac{x^2 + y^2}{\sqrt{1 + 4\,(x^2 + y^2)}}\, dS$$
$$= \iint_{D} x^2 + y^2\, dA = \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{\pi}{2}.$$

5. (20 points) Let

$$\vec{F}(x, y, z) = (e^x - y)\,\vec{i} + (e^y - z)\,\vec{j} + (e^z - x)\,\vec{k}$$

Let $\mathcal{C}$ be the positively oriented triangle which bounds the first octant portion of the plane $\mathcal{S}$ given by

$$x + 2y + z = 2$$

with the normal $\vec{n}$ pointing away from the origin.

(a) (5) Find the *curl* of $\vec{F}$.
Solution: $\vec{\nabla} \times \vec{F} = \vec{i} + \vec{j} + \vec{k}$.

(b) (5) Find the vector $\vec{n}$.
Solution: $\vec{n} = \frac{\vec{i}+2\vec{j}+\vec{k}}{\sqrt{6}}$.

(c) (10) Use Stokes' theorem to find $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ by evaluating a surface integral over $\mathcal{S}$.

Solution:

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \frac{4}{\sqrt{6}}\, dS$$

$$= \int_0^1 \int_0^{2(1-y)} 4\, dx dy = \int_0^1 8(1-y)\, dy = -4(1-y)^2\Big|_0^1 = 4$$

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	7	6		
80-89 (B)	6	10	10		
70-79 (C)	14	13	9		
60-69 (D)	9	7	4		
0-59 (F)	2	0	7		
Cumulative Test Avg	76.9%	78.3%	77.2%	%	%
Cumulative Quiz Avg	72.2%	78.0%	77.7%		