Name:____

Instructions. *Show all work in the space provided: credit is given only for what you write on your paper.* Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading. No books or notes are allowed. A scientific calculator is ok - but not needed . There are **10 (ten)** problems: *maximum total score* = 200.

1. (20 points) Let $f(x, y, z) = e^x \cos y + \ln z$.

   (a) At the point $\left(1, \frac{\pi}{6}, 2\right)$, find a vector $\vec{v}$ pointing in the direction of fastest rate of increase of f.

   (b) Find the directional derivative of f at $\left(1, \frac{\pi}{6}, 2\right)$ in the direction of the vector $\vec{v}$ found in part (a).

2. (20 points) Let $F(x, y, z) = xe^y \cos z - 2z$. Find an equation for the plane that is tangent to the level surface $F(x, y, z) = 1$ at the point $(1, 0, 0)$.

3. (20 points) Use Lagrange multipliers to find the maximum value and the minimum value of $f(x, y, z) = xy^2z$ subject to the constraint that $G(x, y, z) = x^2 + y^2 + z^2 = 1$. (Hint: On the sphere $G(x, y, z) = 1$, f has both positive and negative values. Thus x, y and z are non-zero at the maximum and minimum points, and it is safe to divide by these variables.)

4. (20 points) Find $\iint_D e^{y^2}\, dA$ if $D = \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$.

5. (20 points) Find $\iiint_E x\,dV$ if E is the region bounded by the planes $z = 0$, $z = x$ and $x = 1$ and by the parabolic cylinder $x = y^2$.

6. (20 points) Use spherical coordinates to find the volume of the region R which lies *above* the xy-plane, *inside* the sphere $x^2 + y^2 + z^2 = 1$, and *below* the cone $z = \sqrt{x^2 + y^2}$.

7. (20 points) Let $\vec{F}(x,y,z) = \sin y\vec{i} + (x\cos y + z)\vec{j} + (y+1)\vec{k}$. *Show* how you check that $\vec{F}$ is conservative, *find* a potential function f and *use it to evaluate*

$$\int_{(0,0,1)}^{(1,\frac{\pi}{2},2)} \vec{F} \cdot d\vec{r}$$

8. (20 points) Use Green's theorem to evaluate the line integral

$$\oint_{\mathcal{C}} (\tan(x) - y)dx + (\sec(y) + x)dy$$

if $\mathcal{C}$ is the positively oriented boundary of the triangular region enclosed by the lines

$$y = 0,\ x = 3 \text{ and } y = 2x.$$

9. (20 points)Use Stokes' theorem to find $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ if $\mathcal{C}$ is the positively oriented boundary curve of the part of the surface $\mathcal{S}$ described by

$$z = 1 - x^2 - y^2 \text{ with } z \geq 0$$

and upward unit normal $\vec{n}$, and

$$\vec{F}(x, y, z) = \left(\arctan(x) - yx^2\right) \vec{i} + xy^2 \vec{j} + \arctan(z)\vec{k}.$$

10. (20 points) Use the divergence theorem to find the flux $\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n} \, dS$ if $\vec{F}(x, y, z) = e^{x-y}\vec{i} + e^{x-y}\vec{j} + \left(x^2 + y^2 + z^2\right) \vec{k}$ and $\mathcal{S}$ is the positively oriented surface of the solid bounded by the planes

$$x = 0, \;\; y = 0, \;\; z = 0, \;\; x = 1, \;\; y = 2, \;\text{ and } z = x + y.$$

Solutions

1. (20 points) Let $f(x, y, z) = e^x \cos y + \ln z$.

   (a) At the point $\left(1, \frac{\pi}{6}, 2\right)$, find a vector $\vec{v}$ pointing in the direction of fastest rate of increase of f.
   Solution: $\vec{v} = \vec{\nabla} f \left(1, \frac{\pi}{6}, 2\right) = \left(\frac{e\sqrt{3}}{2}, -\frac{e}{2}, \frac{1}{2}\right)$ or any positive multiple of this vector.

(b) Find the directional derivative of f at $\left(1, \frac{\pi}{6}, 2\right)$ in the direction of the vector $\vec{v}$ found in part (a).

Solution: If $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ then $D_{\vec{u}} f\left(1, \frac{\pi}{6}, 2\right) = |\vec{v}| = \frac{\sqrt{4e^2+1}}{2}$.

2. (20 points) Let $F(x, y, z) = xe^y \cos z - 2z$. Find an equation for the plane that is tangent to the level surface $F(x, y, z) = 1$ at the point $(1, 0, 0)$.

Solution: $\vec{\nabla} F(x, y, z) = e^y \cos z \vec{i} + xe^y \cos z \vec{j} - (xe^y \sin z + 2) \vec{k}$. Thus $\vec{\nabla} F(1, 0, 0) = \vec{i} + \vec{j} - 2\vec{k}$. Hence the tangent plane has the equation $(x - 1) + y - 2z = 0$, or $x + y - 2z = 1$.

3. (20 points) Use Lagrange multipliers to find the maximum value and the minimum value of $f(x, y, z) = xy^2z$ subject to the constraint that $G(x, y, z) = x^2 + y^2 + z^2 = 1$. (Hint: On the sphere $G(x, y, z) = 1$, f has both positive and negative values. Thus x, y and z are non-zero at the maximum and minimum points, and it is safe to divide by these variables.)

Solution: Since the sphere is closed and bounded and f is continuous, the maximum and the minimum exist. We set $\vec{\nabla} f = \lambda \vec{\nabla} G$. This leads to a system of 4 equations in 4 unknowns.

$$\begin{aligned} y^2 z &= 2\lambda x && (1) \\ 2xyz &= 2\lambda y && (2) \\ xy^2 &= 2\lambda z && (3) \\ x^2 + y^2 + z^2 &= 1 && (4) \end{aligned}$$

Dividing (1) by (3) yields $z = \pm x$ and dividing (1) by (2) yields $y = \pm x\sqrt{2}$. Substitution into (4) yields the 8 critical points $\left(\pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}, \pm \frac{1}{2}\right)$. It follows that the maximum value of f is $\frac{1}{8}$ and the minimum is $-\frac{1}{8}$.

4. (20 points) Find $\iint_D e^{y^2}\, dA$ if $D = \{(x, y) \mid 0 \le y \le 1, 0 \le x \le y\}$.

Solution:

$$\iint_D e^{y^2}\, dA = \int_0^1 \int_0^y e^{y^2}\, dx\, dy = \int_0^1 y e^{y^2}\, dy = \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{1}{2}(e - 1)$$

5. (20 points) Find $\iiint_E x\, dV$ if E is the region bounded by the planes $z = 0$, $z = x$ and $x = 1$ and by the parabolic cylinder $x = y^2$.

Solution:

$$\iiint_E x\,dV = \int_{-1}^{1}\int_{y^2}^{1}\int_0^x x\,dz\,dx\,dy = \int_{-1}^{1}\int_{y^2}^{1} x^2\,dx\,dy$$

$$= \frac{1}{3}\int_{-1}^{1} 1 - y^6\,dy = \frac{1}{3}\left(y - \frac{y^7}{7}\right)\Big|_{-1}^{1} = \frac{4}{7}.$$

6. (20 points) Use spherical coordinates to find the volume of the region R which lies *above* the xy-plane, *inside* the sphere $x^2 + y^2 + z^2 = 1$, and *below* the cone $z = \sqrt{x^2 + y^2}$.

   Solution:

$$V = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{2\pi}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi\,d\phi = \frac{\pi\sqrt{2}}{3}.$$

7. (20 points) Let $\vec{F}(x, y, z) = \sin y\vec{\imath} + (x\cos y + z)\vec{\jmath} + (y + 1)\vec{k}$. *Show* how you check that $\vec{F}$ is conservative, *find* a potential function f and *use it to evaluate*

$$\int_{(0,0,1)}^{(1,\frac{\pi}{2},2)} \vec{F}\cdot d\vec{r}$$

   Solution: One can find and check that $f(x, y, z) = x\sin y + (y+1)z$ is a potential function. Thus $\int_{(0,0,1)}^{(1,\frac{\pi}{2},2)} \vec{F}\cdot d\vec{r} = f\left(1, \frac{\pi}{2}, 2\right) - f(0, 0, 1) = \pi + 2$.

8. (20 points) Use Green's theorem to evaluate the line integral

$$\oint_{\mathcal{C}} (\tan(x) - y)dx + (\sec(y) + x)dy$$

   if $\mathcal{C}$ is the positively oriented boundary of the triangular region enclosed by the lines

$$y = 0,\ x = 3 \text{ and } y = 2x.$$

   Solution: $\oint_{\mathcal{C}} (\tan(x) - y)dx + (\sec(y) + x)dy = \int_0^3\int_0^{2x} N_x - M_y\,dy\,dx = \int_0^3\int_0^{2x} 2dy\,dx = 18$.

9. (20 points) Use Stokes' theorem to find $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ if $\mathcal{C}$ is the positively oriented boundary curve of the part of the surface $\mathcal{S}$ described by

$$z = 1 - x^2 - y^2 \text{ with } z \geq 0$$

and upward unit normal $\vec{n}$, and

$$\vec{F}(x, y, z) = \left(\arctan(x) - yx^2\right)\vec{\imath} + xy^2\vec{\jmath} + \arctan(z)\vec{k}.$$

Solution:

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \vec{\nabla} \times \vec{F} \cdot \vec{n}\, dS = \iint_{\mathcal{S}} \left(x^2 + y^2\right)\vec{k} \cdot \frac{2x\vec{\imath} + 2y\vec{\jmath} + \vec{k}}{\sqrt{1 + 4\left(x^2 + y^2\right)}}\, dS$$

$$= \iint_D x^2 + y^2 dA = \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{\pi}{2}.$$

10. (20 points) Use the divergence theorem to find the flux $\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$ if $\vec{F}(x, y, z) = e^{x-y}\vec{\imath} + e^{x-y}\vec{\jmath} + \left(x^2 + y^2 + z^2\right)\vec{k}$ and $\mathcal{S}$ is the positively oriented surface of the solid bounded by the planes

$$x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 2, \text{ and } z = x + y.$$

Solution: $\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = \iiint_B \vec{\nabla} \cdot \vec{F}\, dV = \int_0^1 \int_0^2 \int_0^{x+y} 2z\, dz\, dy\, dx = \int_0^1 \int_0^2 (x+y)^2\, dy\, dx = \int_0^1 \frac{(x+2)^3 - x^3}{3}\, dx = \left.\frac{(x+2)^4 - x^4}{12}\right|_0^1 = \frac{3^4 - 2^4 - 1}{12} = \frac{16}{3}.$

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	7	6	10	8
80-89 (B)	6	10	10	16	14
70-79 (C)	14	13	9	9	13
60-69 (D)	9	7	4	2	2
0-59 (F)	2	0	8	0	0
Cumulative Test Avg	76.9%	77.6%	75.7%	83.7%	81.4% *
Cumulative Quiz Avg	72.2%	78.0%	77.7%	76.3%	76.3%

*Includes replacement of lowest hour test grade by quiz average if higher.