

Print Your Name Here: _____

- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue on the back side.*
- **No** books or notes (paper or electronic) or communication devices (smart/cell phones, internet-connected devices such as laptops, tablets, or I-watches) are allowed. A scientific calculator (*not capable* of graphing or symbolic calculations) is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. **Do not** replace precise answers such as $\sqrt{2}$, $\frac{1}{3}$, or π with decimal approximations. *Keep your eyes on your own paper!*
- The maximum score possible is 100 points.

1. (10 points) *Evaluate* the limit *or show* that it *does not exist* by evaluating along two different paths through the origin: $\lim_{(x,y) \rightarrow (0,0)} \frac{(x+y)^2}{x^2+y^2}$.

2. (20 points) Use the Chain Rule to find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = f\left(\frac{x}{y}\right)$ where f is a differentiable function.

3. (20 points) Let $F(x, y, z) = e^{xyz} - x - y - z$. The point $(0, 0, 1)$ lies on the surface S defined by $F(x, y, z) = 0$.

a. (10) Find the gradient vector $\nabla F(x, y, z)$

b. (5) Find the vector $\nabla F(0, 0, 1)$

c. (5) Find an equation for the tangent plane to the surface S at $(0, 0, 1)$.

4. (20 points) Let $f(x, y) = x^2y$ on the domain $D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$. Find all the absolute maximum points and all the absolute minimum points, and the absolute maximum value and absolute minimum value of f .

5. (30 points) Let $f(x, y) = x^2 + y^2$ and $g(x, y) = xy$.
- a. (10) Find the gradient vectors $\nabla f(x, y)$ and $\nabla g(x, y)$.
- b. (20) Use the method of Lagrange multipliers to find the *absolute extreme values* of $f(x, y)$ on the hyperbola $g(x, y) = 1$. Only one of the two types of absolute extreme values (maximum or minimum) exists. Find the existing absolute extreme value and the point(s) at which that extreme value occurs. Say why the other extreme value does not exist.

Solutions

1. This limit does not exist. The limit along the x -axis as $x \rightarrow 0$ is 1, but the limit along the line $y = -x$ as $x \rightarrow 0$ is 0. This problem was *homework problem* 14.2/15. The work evaluating the two path-dependent limits needs to be correct, but other choices of paths can be made. However, make sure both your paths actually go to the origin! Also, please don't write nonsensical statements about limits. Every part of the calculus is about limits!

2. This application of the Chain Rule was part of problem 14.3/46. $\frac{\partial z}{\partial x} = \frac{f'\left(\frac{x}{y}\right)}{y}$ and $\frac{\partial z}{\partial y} = \frac{-xf'\left(\frac{x}{y}\right)}{y^2}$. Note that $z = f\left(\frac{x}{y}\right)$ depends on the value of $\frac{x}{y}$ and $\frac{x}{y}$ depends on both x and y . Here, f is a function of one real variable, and the ratio $\frac{x}{y}$ depends on two real variables: x and y . Draw a tree diagram if you don't see this.

3. This was part of question 14.6/51.

a. (10) $\nabla F(x, y, z) = \langle yze^{xyz} - 1, xze^{xyz} - 1, xye^{xyz} - 1 \rangle$. The answers to parts (a) and (b) *must be vectors!*

b. (5) $\nabla F(0, 0, 1) = \langle -1, -1, -1 \rangle$

c. (5) $x + y + z = 1$ is an equation for the tangent plane. It is a serious error to omit the equality, which is needed to produce a linear equation satisfied if and only if (x, y, z) lies on the tangent plane. One side of an equation with no other side, across an equal sign, does not specify anything. The equation shown above expresses the orthogonality of ∇F to each vector in the tangent plane at the point of tangency. *It is important not to confuse the method for finding the equation of the tangent plane to an implicitly given surface with the method for an explicitly given surface $z = f(x, y)$! If there was such confusion, you may lose credit even if the final answer turns out (accidentally) to be correct. The implicitly defined surface $F(x, y, z) = 0$ in this problem cannot be easily solved for $z = f(x, y)$. Of course z_x and z_y could be found by the method of implicit differentiation, but these are not the same as F_x and F_y respectively. Some students used the explicit method assuming that $z_x = F_x$ and that $z_y = F_y$, which are false statements, even if they can be true accidentally because of a numerical coincidence.*

4. This problem is very similar to 14.7/38. The absolute extrema *must exist* since f is continuous on a closed, bounded domain D . The absolute minimum value of f is zero, and it occurs at all points of D lying on either coordinate axis. There are no interior extreme points because the two partial derivatives don't vanish there. But one *must always check for interior extreme points!* The

absolute maximum value is $f\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \frac{2}{3\sqrt{3}} = \frac{2}{9}\sqrt{3}$, and this maximum can occur only on

the circular arc that is part of the boundary of D . This can be found by setting $x = \cos\theta$ and $y = \sin\theta$ with $0 < \theta < \frac{\pi}{2}$. Or it could be found using Lagrange multipliers as an alternative method for the *circular part of the boundary curve*. Note also that the second derivative test is *not relevant* to this problem, since you are not seeking local extrema or saddle points. Also, many students went astray because they did not have any idea what the domain D looks like: the first quadrant part of the unit disc!

5. This is 14.8/23

a. $\nabla f(x, y) = \langle 2x, 2y \rangle$ and $\nabla g(x, y) = \langle y, x \rangle$.

- b. f calculates the square of the distance of (x, y) from the origin. For the two branches of the given hyperbola, which are asymptotic to the two axes, that distance has no upper bound, so there is no absolute maximum for f . We seek the absolute minimum points, which are the closest points to the origin. We solve the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ with $g(x, y) = 1$. It is very helpful that neither x nor y nor λ can be 0 since $xy = 1$. So the first two real equations yield $\frac{x}{y} = \frac{y}{x}$ so that $x^2 = y^2$ and also $y = x$ because $xy = 1 > 0$. Thus the closest points to the origin are $(1, 1)$ and $(-1, -1)$ and the minimum value is $f(1, 1) = 2 = f(-1, -1)$. Note that the second derivative test is useless here, since that is a test for local extrema and for saddle points at *interior critical points* of a domain D . Here there are *no interior points*: we are working on two curves: The two branches of a hyperbola! Part (a) was to get you ready to use Lagrange multipliers to find the extrema of f subject to the *constraint* defined by $g(x, y) = 1$. Please note also that ∇f and ∇g are *vector-valued*, *not scalar-valued*!

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4				
80-89 (B)	9				
70-79 (C)	9				
60-69 (D)	4				
0-59 (F)	5				
This Test Avg	75.5%	%	%	%	%
HW Avg	95.1 %	%	%	%	%
HW/TST Corr	0				

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{31}$ —one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have no correlation with performance on the graded homework. In geometrical terms, the test vector is orthogonal to the graded homework vector. So let me ask: Does everyone have a notebook with *all the assigned problems—not just the collected problems—solved correctly*? You can see that every test question came directly from the assigned homework.