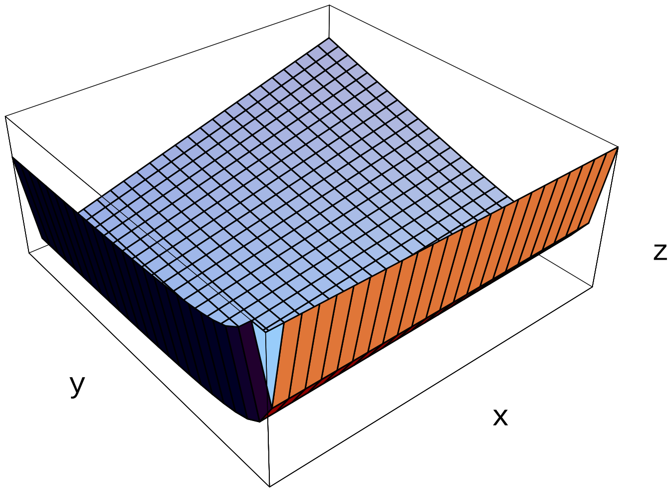

If we try to minimize the surface area of an open-topped rectangular box of volume 48 with edges x, y, z, we are led to try to minimize the function $f(x,y) = xy + \frac{96}{x} + \frac{96}{y}$, with $x > 0$ and $y > 0$. Observe that the surface approaches infinite height near either the x or the y-axis. Also, for each fixed y, $f(x,y) \to \infty$ as $x \to \infty$, though quite slowly if y is small. There is similar behavior as $y \to \infty$ for fixed x. As x and $y \to \infty$ $f(x,y)$ rises quickly towards ∞. Thus we can see intuitively that this function, which is differentiable everywhere on its domain of definition, has a graph which is a surface hanging down to a minimum value at some modest distance from the origin, and the minimum must be achieved at a point where *both* $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

The intuitive analysis above depends upon our visual sense of how the surface hangs above the first quadrant of the xy-plane. This is sufficient for Math 2057. However, for the curious, let us give now a careful, rigorous analysis of why the minimum must exist as claimed above by intuitive examination of the graph. *(What follows is not required for Math 2057.)*

First, pick any convenient value of f, say $f(1,1) = 193$. Let $M > 193$. Consider the arc in the first quadrant of the hyperbola $xy = M$. Above that arc, $f(x,y) \geq M > f(1,1)$. However, below that arc, if $x > c = \frac{M^2}{96}$, then $y < \frac{M}{c}$, so that $f(x,y) \geq \frac{96}{y} > \frac{96c}{M} = M > f(1,1)$. Similar reasoning applies if $y > c = \frac{M^2}{96}$. Also, if either $x < \frac{96}{M}$ or $y < \frac{96}{M}$, then $f(x,y) > M$. Thus if we let D denote the domain for which $xy \leq M$ with $\frac{96}{M} \leq x \leq c$ and $\frac{96}{M} \leq y \leq c$, then D is a closed bounded domain, and f has an absolute minimum value μ on D. But then $\mu \leq f(1,1) < f(x,y)$ for all (x,y) outside D. Hence μ is actually the absolute minimum value of f on the whole first quadrant of the xy-plane.