Print Your Name Here: ______________________________

- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. Please do ***not*** replace precise answers with decimal approximations.
- There are **eight (8)** problems: *Maximum total score* = 200.

1. (25) For the linear system

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 &= 0 \\ 4x_1 + 3x_2 - x_3 &= 0 \end{aligned}$$

write the *augmented matrix* A, *row-reduce* that matrix to rref(A), and *find all solutions* to the system. Find a *basis* for $\ker(A)$.

2. (25) Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Determine whether or not A is invertible, and if it is invertible, find A^{-1}. (Suggestion: check your work by multiplying.)

3. (25) Consider the linear system represented by the augmented matrix $A = \left[\begin{array}{ccc|c} a & 4 & 3 & 1 \\ 0 & d & f & 2 \\ 0 & 0 & f & 0 \end{array}\right]$. In each of the following cases, how many solutions does the system have?

a. (9) a and d are nonzero but $f = 0$.

b. (8) a and d and f are all nonzero.

c. (8) a and f are nonzero but $d = 0$.

4. (25) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a *unit vector* and let $\mathfrak{S} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the *standard basis* in $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = \text{proj}_L(\mathbf{x})$ where L is the straight line containing the vector $\mathbf{u}$ and where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is any vector in $\mathbb{R}^2$. Find:

a. (7) $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ in terms of u_1 and u_2.

b. (7) The matrix $[T]$ in terms of u_1 and u_2 using the standard basis of $\mathbb{R}^2$.

c. (6) The sum of the diagonal entries of $[T]$ in part (b). (Give a numerical result.)

d. (5) Find the matrix of the *reflection* $[\text{ref}_L]$ in terms of u_1 and u_2. (Hint: To express $\text{ref}_L(\mathbf{x})$ in terms of $\text{proj}_L(\mathbf{x})$ one can sketch a parallelogram with L as the diagonal, $\mathbf{x}$ as one side and $\text{ref}_L(\mathbf{x})$ as the other side.)

5. (25) Find a *basis* $\mathfrak{B}$ for the vector space V of all $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and *find the dimension* of V.

6. (15) Let $T : P_2 \to P_2$ by $T(f)(x) = 2f''(x) - f(x)$. Use $\mathfrak{B} = \{1, x, x^2\}$ for a basis of P_2.

a. Find the matrix $[T]_{\mathfrak{B}}$ of T using the basis $\mathfrak{B}$.

b. Find $\det(T)$.

c. Is T an isomorphism? Why or why not?

7. (25) Let $T : P_2 \to P_2$ by $(Tf)(t) = (t-1)f'(t)$. Let $B = \{t^2, t, 1\}$ and $B' = \{(t-1)^2, t-1, 1\}$ be two bases for the vector space P_2 of polynomials of degree at most two.

a. (7) Find the matrix $[T]_B$ for T in the basis B. Is T an *isomorphism*? *Why or why not*?

b. (7) Find the matrix $[T]_{B'}$ for T in the basis B'.

c. (7) Find the *change of basis matrix* $S_{B' \to B}$.

d. (4) Calculate the two products $[T]_B S_{B' \to B}$ and $S_{B' \to B}[T]_{B'}$. What do you know must be true about these two products?

8. (35) Use *row-reduction* to calculate the value of $\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 65 \end{bmatrix}$.

Solutions

1. $A = \left[\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 4 & 3 & -1 & 0 \end{array}\right] \to \text{rref}(A) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$. Thus $x_1 = -\frac{t}{2}$, $x_2 = t$, $x_3 = t$ where $t \in \mathbb{R}$. A basis for $\ker(A)$ is $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \right\}$ Any nonzero constant multiple of this basis vector would also be correct. It would have helped some students to check that their "solutions" actually satisfy the pair of linear equations. Compare with 1.2/#2.

2. A is invertible, and $A^{-1} = \dfrac{1}{9}\begin{bmatrix} -2 & 1 & 4 \\ 1 & 4 & -2 \\ 4 & -2 & 1 \end{bmatrix}$. Compare with 2.4/#5.

3. Compare with 1.3/1.

a. There are infinitely many solutions. Note that rref(A) has no leading one in the third column.

b. There is a unique solution.

c. Subract row 3 from row 2 and the second equation cannot be satisfied so there are no solutions.

4. See 2.2/13. It is important to know what is meant by the *standard basis* for $\mathbb{R}^2$. One needs to understand that $\mathbf{e}_1$ is a vector and $\mathbf{e}_2$ is a vector and $\mathbf{u}$ is a *unit vector*.

a. $T(\mathbf{e}_1) = \begin{bmatrix} u_1^2 \\ u_1u_2 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} u_1u_2 \\ u_2^2 \end{bmatrix}$.

b. $[T] = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$.

c. 1

d. $[\text{ref}_L] = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix} = \begin{bmatrix} u_1^2 - u_2^2 & 2u_1u_2 \\ 2u_1u_2 & u_2^2 - u_1^2 \end{bmatrix}$.

5. $\mathfrak{B} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ and dim(V)=2. Any linearly independent pair of 2×2 matrices satisfying $b = -a$ and $d = -c$ will suffice for a basis. See 4.1/30.

6.

a. $[T]_{\mathfrak{B}} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

b. $\det(T) = -1$.

c. T is an isomorphism because the reduced row-echelon form of the matrix of T is I_3, so that T it is one-to-one, or alternatively since $\det T \neq 0$. This makes $N(T) = 0$ and $R(T) = 3$ so that T is onto.

7. Compare with 4.2/45 and 4.3/27, 28, 47.

a. $[T]_B = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$. For example, column one of the matrix is the coefficient vector of $T(t^2) = (t-1)2t = 2t^2 - 2t$ and that coefficient vector is the first column shown for the matrix: ie twice the first basis vector minus twice the second basis vector. T is *not* an isomorphism: it maps the third basis vector to 0 so the kernel is not just the zero vector and T is *not one-to-one*. Also, T is *not onto* P_2 since its rank is two.

b. $[T]_{B'} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The concept of the matrix of a linear transformation with respect to a given basis is one of the most fundamental concepts of this course. In every example the jth column is T of the jth basis vector written as a coordinate vector in the given basis, which is B' in this example.

c. $S_{B' \to B} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. The jth column is the coordinate vector of the jth vector in B' with respect to the basis B.

d. Both products equal $\begin{bmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}$. One knows even before the matrices are multiplied that the two products must be equal.

8. Row reduction to an upper triangular matrix shows that $\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 65 \end{bmatrix} = -4$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	14	6	10	11
80-89 (B)	6	4	4	8	8
70-79 (C)	3	3	4	2	3
60-69 (D)	5	0	6	4	3
0-59 (F)	8	5	6	3	2
Test Avg	74%	84%	72.7%	81.4%	81.79%
Cumulative HW Avg	79.5%	80.1%	80.6%	80.9%	80.9%
HW/Test Correl	–	0.84	0.81	0.72	0.72

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{27}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.