Print Your Name Here: ____________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Suppose $s_n \le t_n \le u_n$ for all n, $s_n \to a < b$, and $u_n \to b$. True or give a counterexample: $\lim_{n\to\infty} t_n \in [a, b]$.

2. True or give a counterexample: If $x_n \in \mathbb{R}$ is a Cauchy sequence, then every subsequence x_{n_k} must be Cauchy.

3. Let $x_n = (1.5)^n$, for all $n \in \mathbb{N}$ and let T_n be the nth *tail* of the sequence. Find:

a. (1 point) $\sup(T_n) =$

b. (1 point) $\inf(T_n) =$

c. (2 points) $\liminf x_n =$

d. (2 points) $\limsup x_n =$

4. Suppose for all $n \in \mathbb{N}$ we have $y_n \neq 0$. Prove or else give a counterexample: If both $x_n y_n$ and $\frac{x_n}{y_n}$ converge, then x_n converges and y_n converges.

5. Give an example of a decreasing nest of nonempty open finite intervals $(a_1, b_1) \supseteq (a_2, b_2) \supseteq \cdots$ such that $\bigcap_{k=1}^{\infty}(a_k, b_k) = \emptyset$, the *empty set*.

6. True or Give a Counterexample: A rational number times $\sqrt{2}$ is irrational.

7. True or give a Counterexample: If a sequence is unbounded, then every subsequence is unbounded.

8. True or give a Counterexample: If a sequence does not have a smallest element, then it has a largest element.

9. True or Give a Counterexample: If x_n diverges to infinity, then every subsequence diverges to infinity.

10. Let $E \subseteq \mathbb{R}$ be any *unbounded* set. Find an open cover $\mathcal{O} = \{O_n \mid O_n \text{ is open } \forall n \in \mathbb{N}\}$ of E that has no finite subcover.

11. Find the set of all cluster points of the open interval (a, b), where $a < b$.

12. True or give a counterexample: Every open cover of a finite subset of $\mathbb{R}$ has a finite subcover.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Suppose $a \leq x_n \leq b$ for all $n \in \mathbb{N}$ and suppose further that $x_n \to L$. Prove: $L \in [a, b]$. (Hint: Prove there is a contradiction if $L < a$, and also if $L > b$.)

B. Suppose A and B are subsets of $\mathbb{R}$, both nonempty, with the special property that $a \leq b$ for all $a \in A$ and for all $b \in B$. Prove: $\sup(A) \leq \inf(B)$. (Hint: Every b is an upper bound of A. So how does the $\sup(A)$ relate to each $b \in B$?)

C. Let $E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. Find an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover. Be sure to prove that $\mathcal{O}$ is an open cover and that $\mathcal{O}$ has no finite subcover.

Solutions and Class Statistics

1. Counterexample: $s_n \equiv -1 \leq t_n = (-1)^n \leq u_n \equiv 1$, so that $\lim_{n\to\infty} t_n$ does not exist.

2. True, since every subsequence of a convergent sequence must converge (and indeed to the same limit).

3.

a. $\sup(T_n) = \infty$

b. $\inf(T_n) = 1.5^n$

c. (2 points) $\liminf x_n = \infty$

d. (2 points) $\limsup x_n = \infty$

4. Sorry–typo: this should have said True or give a Counterexample. Proofs are not required for short questions. Counterexample: Let $x_n = (-1)^n = y_n$.

5. For example, let $(a_k, b_k) = \left(0, \frac{1}{k}\right)$ for all $k \in \mathbb{N}$.

6. Counterexample: $0\sqrt{2}$ is rational.

7. Counterexample: Let x_n be 0 if n is odd, and n if n is even. Caution: $\mathbb{R}$ is *not* a sequence.

8. Counterexample: $x_n = (-1)^n n$. There are infinitely many such counterexamples. Caution: $\mathbb{R}$ is *not* a sequence. Note: In a constant sequence each term is both the largest and smallest.

9. True.

10. For example, let $O_n = (-n, n)$ for all $n \in \mathbb{N}$.

11. $[a, b]$.

12. True

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: Inequalities are the nuts and bolts of analysis. They must be correct. We are given only that $x_n \in [a, b]$ for all $n \in \mathbb{N}$ and that $x_n \to L$ as $n \to \infty$. If $L < a$ or if $L > b$, we need $N \in \mathbb{N}$ such that

$n \geq N$ implies that $x_n \notin [a,b]$. In either case, $L < a$ or $L > b$, the key is to pick $\epsilon > 0$ correctly so as as to force the existence of an $N \in \mathbb{N}$ such that $n \geq N \implies x_n \notin [a,b]$. Be very clear about the fact that it is *after* we pick $\epsilon > 0$ that there exists an $N \in \mathbb{N}$, corresponding to the choice of $\epsilon > 0$, such that $n \geq N \implies |x_n - L| < \epsilon$. This is the kind of reasoning you have been practicing in the homework.

B: Explain that A is bounded above (by each $b \in B$) and that B is bounded below (by each $a \in A$). Since each $b \in B$ is an upper bound of A, how must $\sup(A)$ relate to B. Then explain how $\sup(A)$ relates to $\inf(B)$.

C: Make sure your open cover is a collection of open sets, and that E is contained in the union of that collection of sets. Let F be a finite subset of your index set $\mathbb{N}$ and prove that the union of the $O_n,\ n \in F$ fails to cover E.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7				
80-89 (B)	7				
70-79 (C)	8				
60-69 (D)	2				
0-59 (F)	6				
Test Avg	75.6%	%	%	%	%
HW Avg	7.2				
HW/Test Correl	0.79		–		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{32}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.