I have read, understood, and complied with the instructions in the box below. Legible

***Signature* and LSU ID #:** ______________________

- Download and print a copy of this test on 8.5 by 11 inch letter size paper. If no printer is available, make a hand-written facsimile. Be sure to sign the statement above.
- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue a problem on a second page.*
- ***Books/notes (electronic/paper), cell/smart phones, computers and all internet-connected devices are prohibited!*** A scientific calculator (*not capable* of graphing or symbolic calculations) is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. *Do not replace* precise answers, such as $\sqrt{2}$, π, or $\cos\frac{\pi}{7}$ with decimal approximations. *Make all obvious simplifications.* Submit only your own work!
- Because this is a take-home test on an *honor system*, you have **90 minutes** to complete this hour-test, instead of the usual 50 minutes. Start counting time *after the test is printed* and you are ready to begin the mathematical work. *At the end of the 90 minutes, take extra time* to make a scan file or camera copy of your work (pdf file containing all the pages preferred) and email that file to me **rich@math.lsu.edu** as soon as possible but no later than 6 PM Monday April 13, 2020. *These instructions express my trust and confidence in your integrity and good character.*

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: If A_n is a countable set for each $n \in \mathbb{N}$, then $\mathcal{A} = \bigcup_{n=1}^{\infty} A_n$ is also a countable set.

2. True or False: An *open, dense* subset of $\mathbb{R}$ must be all of $\mathbb{R}$.

3. Find all cluster points of the set $I = \mathbb{R} \setminus \mathbb{Q}$ of irrational numbers.

4. Let $f : \mathbb{Z} \to \mathbb{R}$[1]. Find all points of *continuity* of f.

5. True or False: If $f(x) \to 0$ as $x \to a$ then $\dfrac{1}{f(x)} \to \infty$ as $x \to a$.

6. True or False: If $f(x) \to \infty$ as $x \to a$ then $\dfrac{1}{f(x)} \to 0$ as $x \to a$.

7. Let $f(x) = \begin{cases} 1-x & \text{if } x \in \mathbb{Q}, \\ 1-x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find all points of continuity of f.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be any *continuous* function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f(2) = 1$, find $f(-3)$.

9. Give and example of a function f and a domain D such that $f \in \mathcal{C}(D)$ but f is *not* uniformly continuous on D.

10. Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ -x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find $\|f\|_{\text{sup}}$.

[1] $\mathbb{Z}$ is the set of all integers.

11. Suppose $f_n \in \mathcal{C}[0,1]$ for all $n \in \mathbb{N}$ and suppose for each fixed $x \in [0,1]$, $f_n(x)$ is a sequence decreasing monotonically to the limit $f(x) = x$. True or False: f_n converges uniformly on $[0,1]$.

12. Let $f_n(x) = xe^{-nx}$ for all $n \in \mathbb{N}$ and all $x \in [0,\infty)$. Find $\|f_n\|_{\sup}$ for all $n \in \mathbb{N}$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Prove: If f is *monotone increasing*[2] on $\mathbb{R}$, then for all $a \in \mathbb{R}$, $\lim_{x\to a+} f(x)$ exists and is a real number. The latter limit is called the *limit from the right*, and it is denoted by $f(a+)$. (Hint: Let $S = \{f(x) \mid x > a\}$ and show that $\inf(S)$ is a real number L. Then show that for every sequence $x_n \to a+$ we must have $f(x_n) \to L$.)

B. Prove the following *fixed point theorem*: Suppose $f \in \mathcal{C}[0,1]$ and suppose $0 \le f(x) \le 1$ for all $x \in [0,1]$. Then there exists $c \in [0,1]$ such that $f(c) = c$. (The point c is then called a *fixed point* for the function f. Hint: Consider $g(x) = f(x) - x$ and use the intermediate value theorem.)

C. Let $0 \le b < 1$ and let $f_n(x) = 1 - x^n$ for all $x \in [0,1)$.

- **(i)** Find the pointwise limit $f(x)$ of the sequence $f_n(x)$ for all $x \in [0,1)$.
- **(ii)** Let b be an arbitrary number in $[0,1)$. Prove that f_n converges *uniformly on* $[0,b]$.
- **(iii)** Determine whether or not f_n converges uniformly on $[0,1)$ and prove your conclusion.

[2] That is, if $x_1 < x_2$ then $f(x_1) \le f(x_2)$.