Print Your Name Here: ______________________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: Every subset of $\mathbb{R}$ can be expressed as the union of a family of *closed* sets.

2. True or give a counter-example: Every *infinite* subset of the set of *irrational* numbers is *uncountable*.

3. True or give a counter-example: A *dense* open subset of $\mathbb{R}$ must be the the whole set $\mathbb{R}$.

4. Find the *set of all cluster points* of the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers.

5. Give an example of a monotone increasing function f defined on $\mathbb{R}$ such that $\lim_{x\to 0+} f(x) \neq f(0)$.

6. Let $f(x) = \begin{cases} \frac{x^7-1}{x-1} & \text{if } x \neq 1, \\ c & \text{if } x = 1, \end{cases}$. Find the value of c that makes $f \in \mathcal{C}(\mathbb{R})$.

7. Let $f \in \mathcal{C}(\mathbb{R})$ such that $f(x+y) \equiv f(x) + f(y)$. If $f(2) = 5$, find $f(7)$.

8. Let $Q(x) = \sqrt{x}$ for all $x \geq 0$. Find a value of $\delta > 0$ such that $a, x \in [0, \infty)$ and $|x - a| < \delta$ implies $|Q(x) - Q(a)| < \epsilon$.

9. True or False: If f is uniformly continuous on $(-1, 0]$ and also on $[0, 1)$ then f is uniformly continuous on $(-1, 1)$.

10. True or give a counter-example: If f is uniformly continuous on the domain $D_1 = (-1, 0]$ and also on the domain $D_2 = (0, 1)$ then f is uniformly continuous on $D_1 \cup D_2 = (-1, 1)$.

11. Give an example of a function $f \in \mathcal{C}(\mathbb{R})$ such that f is uniformly continuous on $[0, 1000]$ but is not uniformly continuous on $[0, \infty)$.

12. Let $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$

a. True or False: $f \notin \mathcal{C}[0, 1]$ but does have the *intermediate value property* on $[0, 1]$.

b. True or False: $f \in \mathcal{C}(0, 1]$ but f is *not uniformly* continuous on $(0, 1]$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. A subset $E \subset \mathbb{R}$ is called *closed* if and only if its complement $\mathbb{R} \setminus E$ is *open.* (For example, $\mathbb{R}$ itself is a closed set since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is an open set.) *Prove* that a closed set E that is also *dense* in $\mathbb{R}$ must be all of $\mathbb{R}$. (Hint: Suppose the claim were false, so that $\mathbb{R} \setminus E$ is a nonempty open set. Deduce a contradiction.)

B. Prove: If $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$ then $\lim_{x \to 0+} f(x)$ does not exist.

C. Let $f(x) = \begin{cases} 1 - x & \text{if } x \in \mathbb{Q}, \\ 1 - x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Prove that f is continuous at p *if* and *only if* $p \in \{0, 1\}$. (Hint: Remember to prove both directions: *if* and also *only if*.)

Solutions and Class Statistics

1. True: Every subset E of $\mathbb{R}$ can be expressed as $E = \bigcup_{x \in E} \{x\}$. Problem 1.94.

2. Counter-example: $\{n\sqrt{2} \mid n \in \mathbb{N}\}$. There are many other examples.

3. Counter-example: If $\mathbb{Q} = \{r_n \mid n \in \mathbb{N}\}$, let $\mathcal{O} = \bigcup_{n \in \mathbb{N}} \left(r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n} \right)$. See Example 1.15, page 33. There are many simpler examples, such as $\mathbb{R} \setminus \{0\}$.

4. $\mathbb{R}$.

5. For example: let $f(x) = 1_{(0,\infty)}$, so $\lim_{x \to 0+} f(x) = 1 \neq f(0) = 0$.

6. $c = 7$.

7. $f(1) = \frac{5}{2}$, so $f(7) = \frac{35}{2}$. See Exercise 2.27.

8. $\delta \leq \epsilon^2$ will work. See Exercises 2.21 and 2.46.

9. True: See homework problem 2.40.

10. Counter-example: Let $f(x) = \begin{cases} 1 \text{ if } 0 < x < 1 \\ 0 \text{ if } -1 < x \leq 0 \end{cases}$.

11. For example, $f(x) = x^2$. See Exercise 2.43.

12. True for both parts. See Exercise 2.47.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: Most students who chose this problem did very well. It is just a matter of knowing the meaning of $R \setminus E$ being open and E being dense. Then the indirect proof follows naturally.

B: Most of the errors in proof B were either wrong trigonometry or misstatement of the Sequential Criterion for Limits of Functions. Always make sure that what you write makes good logical sense.

C: To prove that continuity at p implies $p \in \{0, 1\}$, use the density of both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ to prove that p must be either 0 or 1. To prove continuity at those two points, it is convenient to define two polynomials $g(x) = 1 - x$ and $h(x) = 1 - x^2$ for all $x \in \mathbb{R}$. For p equal to either 0 or 1 establish two values of δ (for g and for h) as explained in class, and show that the minimum of those two deltas works for f regardless of whether x is rational or irrational. Hand waving does not suffice.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	12			
80-89 (B)	7	6			
70-79 (C)	8	3			
60-69 (D)	2	3			
0-59 (F)	6	5			
Test Avg	75.6%	79.8%	%	%	%
HW Avg	7.2	7.62			
HW/Test Correl	0.79	0.66			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{29}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.