Print Your Name Here: ______________________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Give an example of a decreasing nest of open intervals $(a_1, b_1) \supseteq (a_2, b_2) \supseteq \cdots$ such that $b_k - a_k \to 0$ yet $\bigcap_{k=1}^{\infty}(a_k, b_k) \neq \emptyset$.

2. True or False: the set $S = \left\{ \frac{m}{n}\sqrt{2} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$ of all rational multiples of the square root of 2 is a *dense* subset of $\mathbb{R}$.

3. Give an example of an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ that has no finite subcover.

4. True or False: An *open* dense subset of $\mathbb{R}$ must be all of $\mathbb{R}$.

5. True or False: An *open dense* subset of $\mathbb{R}$ must be *uncountable.*

6. Give an example of a sequence of real numbers $x_n \to 0+$ such that $\sin \frac{1}{x_n} = (-1)^{n+1}$.

7. Give an example of a monotone *increasing* function $f(x)$ for which $\lim_{x\to 0+} f(x) > f(0)$.

8. Find all points p at which $a(x) = |x|$ is continuous.

9. Let $f(x) = \begin{cases} 1+x & \text{if } x \in \mathbb{Q}, \\ 1-x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find all points p at which f is continuous.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function such that $f(x+y) \equiv f(x) + f(y)$. If $f(3) = 12$ find $f(\sqrt{2})$.

11. Give an example of an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of $S = \left\{-\frac{1}{n} \mid n \in \mathbb{N}\right\}$ that has no finite subcover.

12. Let $f(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{if } x \neq 2, \\ c & \text{if } x = 2. \end{cases}$ Find the value of c that makes f continuous on $\mathbb{R}$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $E \subset \mathbb{R}$ be any set with the property that there is a *cluster point* p of E such that $p \notin E$. Describe an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover. Justify that $\mathcal{O}$ is an open cover and that it has no finite subcover of E.

B. Suppose $f(x) \leq g(x) \leq h(x)$ for all $x \in (a-\delta, a+\delta) \setminus \{a\}$. Suppose also that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} h(x)$ both exist and equal $L \in \mathbb{R}$. Prove that $\lim_{x \to a} g(x)$ exists and equals L. (This statement is sometimes called the *squeeze theorem* for functions.)

C. Let $Q(x) = \sqrt{x}$, which is defined for all $x \geq 0$. Prove: $Q \in \mathcal{C}[0, \infty)$. (Hint: If $a \geq 0$, and $\epsilon > 0$, we seek $\delta > 0$ such that $x \geq 0$ and $|x - a| < \delta$ implies $|Q(x) - Q(a)| < \epsilon$. Begin by showing that $|\sqrt{x} - \sqrt{a}|^2 \leq |x - a|$.)

Solutions and Class Statistics

1. For example, $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\} \neq \emptyset$

2. True.

3. Let $O_n = (-n, n)$ for each $n \in \mathbb{N}$ for example.

4. False: see the construction of a very small open dense subset of $\mathbb{R}$.

5. True. It must be a nonempty open set and so it is a union of some family of open intervals. Each open interval is uncountable.

6. For example, let $x_n = \dfrac{2}{(2n+3)\pi}$.

7. For example, let $f(x) = 1_{(0,\infty)}(x)$.

8. $\mathbb{R}$.

9. $p = -1$ and $p = 0$.

10. Since $f(x) = cx$ with $c = 4$, we must have $f(\sqrt{2}) = 4\sqrt{2}$.

11. For example, let $O_n = \left(-2, -\dfrac{1}{n}\right)$ for all $n \in \mathbb{N}$.

12. $c = 12$

Comments regarding proofs: In each proof, you should make sure of the following things. Have you made it clear how you are using whatever theorems need to be invoked to complete your argument? Have you taken care not to use undefined terms? It is your task to convince me that you know why the conclusions follow from the hypotheses. Do not leave it to me to fill in your missing reasons

A: Remember that all we know about E is that p is a cluster point of E but $p \notin E$. Create an open cover of $\mathbb{R} \setminus \{p\}$ which does not cover p, so that this cover also covers E. Use the fact that p is a cluster point of E to prove that there does not exist a finite subcover of E. See your notes about this homework problem.

B: Make use of both directions of implication in the sequential criterion for limits of functions, and make use of the squeeze theorem for sequences. Alternatively, make use of the definition of limits of functions , being careful about the domain on which the given inequality is true. In either case, do not make claims about $\lim_{x \to a} g(x)$ before you have proven that this limit exists!

C: I have seen some very bizarre attempts to prove the initial inequality. Please check your class notes to see how simple it is! If I marked your proof of the inequality as wrong or lacking, please convince me otherwise.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4	4			
80-89 (B)	6	4			
70-79 (C)	2	4			
60-69 (D)	1	0			
0-59 (F)	2	1			
Test Avg	80.9 %	82.8%	%	%	%
HW Avg	8.5	71.8			
HW/Test Correl	0.76	0.56			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{13}$–one dimension for each student enrolled and engaged in the course. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a good positive correlation with performance on the homework.