Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Let $p(x) = a_{2n}x^{2n} + \cdots + a_1x + a_0$ be any polynomial of *even degree.* True or Give a Counterexample: If $a_{2n} < 0$, then p has a *maximum* value on $\mathbb{R}$.

2. For each $n \in \mathbb{N}$, let X_n be the set of polynomials of degree equal to n. True or give a Counterexample: The set X_n is a vector space.

3. For each $n \in \mathbb{N}$, let P_n be the set of all polynomials of degree less than or equal to n. True or give a Counterexample: The set P_n a vector space.

4. Find $\|f\|_{\sup}$ if: $f(x) = x$ on $\left(-1, \frac{1}{2}\right)$.

5. $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ -x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find $\|f\|_{\sup}$.

6. Let $f_n(x) = xe^{-nx^2}$ for all $x \in \mathbb{R}$. You may use derivatives to answer the following questions.

a. Find $\|f_n\|_{\text{sup}}$.

b. On $\mathbb{R}$ does f_n converge *pointwise*, *uniformly*, *both*, or *neither*?

7. If $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$ find $\overline{\int_0^1} f$ and $\underline{\int_0^1} f$, the upper and lower integrals.

8. Give an example of a *bounded* function on the interval $[0, 1]$ that is not Riemann integrable.

9. True or False: If $f(x)$ is 0 on the interval [0,1] and 2 on the interval (1,2], then there is an $\bar{x} \in [0, 2]$ for which $\int_0^2 f(x)\,dx = 2f(\bar{x})$.

10. Express $\displaystyle\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \cos\left(1 + \frac{2k}{n}\right)$ as an integral. (Be sure to include the lower and upper limits of integration.)

11. Suppose $f : [0, 3] \to \mathbb{R}$ by letting $f(x)$ be 1 if $x \in \{1, 2\}$ and 0 otherwise. If $\epsilon > 0$, find a $\delta > 0$ for which $\|P\| < \delta$ implies $U(f, P) - L(f, P) < \epsilon$.

12. True or Give a Counterexample: If $|f|$ is Riemann integrable on $[a, b]$, then $f \in R[a, b]$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Suppose $f_n(x) = 1 - x^n$. Decide whether or not f_n converges uniformly on each interval, and prove your conclusion.

- **(i)** $[0, 1]$
- **(ii)** $[0, b]$, where $0 \leq b < 1$
- **(iii)** $[0, 1)$

B. *Use the Bolzano-Weierstrass Theorem* to give an alternative proof that $f \in \mathcal{C}[a, b]$ implies that f is bounded. (Hint: Suppose false, so there exists $f \in \mathcal{C}[a, b]$ for which $|f|$ is not bounded above. For all $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Now deduce a contradiction of the fact that $f \in \mathcal{C}[a, b]$.)

C. Let $f \in \mathcal{C}[a, b]$. Prove: There exists $\bar{x} \in [a, b]$ such that $\displaystyle\int_a^b f(x)\,dx = f(\bar{x})(b - a)$. (Hint: Let $m = \inf\{f(x) \mid x \in [a, b]\}$ and $M = \sup\{f(x) \mid x \in [a, b]\}$.

Solutions and Class Statistics

1. True.

2. Counterexample: X_1 is not a vector space, since $0 \cdot x$ has degree 0. However, note that X_0 is a one-dimensional vector space.

3. True.

4. $\|f\|_{\sup} = 1$

5. $\|f\|_{\sup} = \infty$

6.

a. $\|f_n - 0\|_{\sup} = \dfrac{1}{\sqrt{2en}}$.

b. On $\mathbb{R}$ f_n converges *both* pointwise and uniformly (to zero).

7. $\overline{\int_0^1} f = 1$ and $\underline{\int_0^1} f = -1$

8. The indicator function of the rational numbers in $[0,1]$ is an example.

9. False.

10. $\displaystyle \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \cos\left(1 + \frac{2k}{n}\right) = \int_1^3 \cos x \, dx = \sin 3 - \sin 1$. It is sufficient to write the definite integral without anti-differentiating.

11. Any positive $\delta \leq \frac{\epsilon}{4}$ will suffice.

12. Counterexample: Let $f : [0,1] \to \mathbb{R}$ by letting $f(x)$ be 1 if x is rational and -1 if x is irrational. Notice that the upper and lower integrals of f are different.

Remarks about the proofs

Proofs are graded for logical coherence. The sequence of steps should be logical, and *each claim should be justified explicitly* to show that you understand your own proof. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs.

In proof (A), for all three parts, one needs to know *first* the *pointwise* limit $f(x)$ on $[0,1]$ of $f_n(x)$ as $n \to \infty$. For part (i), the discontinuity of f on [0,1] settles the matter. For (ii) one needs *first* to calculate the sup-norm of $f - f_n$ on $[0,b]$ and *then* calculate the limit of this norm as $n \to \infty$. For part (iii) one needs to show that on $[0,1)$ $\|f_n - f\|_{\sup} = 1$ for each fixed n, and then the constant sequence 1 does not converge to 0 as $n \to \infty$. The *order* in which one finds the sup-norm or finds the limit as $n \to \infty$ is critical.

In proof (B), the key is to get a convergent *subsequence* of x_n from the Bolzano-Weierstrass theorem. Then the subsequence converges to a point p that *must* be in $[a, b]$, which forces f to be continuous at p. Use the continuity of f at p to show that $f(x_{n_j})$ must be bounded and explain why this is a contradiction of the construction of x_n. Most of the mistakes were statements that were careless regarding these matters.

In proof (C), it is essential to use the Extreme Value Theorem to show that m and M are values acheived by $f(x)$ at points lying in $[a, b]$. Then use the fact that the Riemann integral preserves inequalities along with the Intermediate Value Theorem to show that $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a value achieved by f at a point $\bar{x} \in [a, b]$.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	6	6	6		
80-89 (B)	2	2	4		
70-79 (C)	5	5	4		
60-69 (D)	1	3	2		
0-59 (F)	7	2	3		
Test Avg	73.5%	78.4%	79.9%	%	%
HW Avg	8.26	7.8	7.5		
HW/Test Correl	0.90	0.83	0.77		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{20}$– one dimension for each student currently enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.