Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Yes or No: Is $f(x) = \dfrac{1}{x}$ *uniformly* continuous on:

a. (0,1)?

b. $[1, 2]$?

c. $(1, \infty)$?

2. True or False: Suppose $f \in \mathcal{C}[0, 1]$ and suppose $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. Then there exists $c \in [0, 1]$ such that $f(c) = c^2$.

3. Let $f(x) = 1000 + 100x^2 + \frac{1}{1000}x^3$. True or False: the equation $f(x) = 0$ has no real root.

4. For each $n \in \mathbb{N} \cup \{0\}$, let X_n be the set of polynomials of degree *equal* to n, and let $P_n = \displaystyle\bigcup_{k=0}^{n} X_k$.

True or *Give a Counterexample:*

a. The set X_n is a vector space for each $n \in \mathbb{N} \cup \{0\}$.

b. The set P_n a vector space for each $n \in \mathbb{N} \cup \{0\}$.

5. Find $\|f\|_{\sup}$ if

a. $f(x) = x$ on $\left(-1, \frac{1}{2}\right)$.

b. $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ -x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$

6. On each of the following intervals, *Answer Yes or No:* Does $f_n(x) = \frac{x}{x+n}$ converges *uniformly* on:

a. $[0, 1]$?

b. $[0, \infty)$?

7. Let $f_n(x) = \frac{x}{1+nx^2}$ for all $x \in \mathbb{R}$.

a. Find the *pointwise* limit of $f_n(x)$ for all $x \in \mathbb{R}$.

b. Does f_n converge uniformly on $\mathbb{R}$?

8. Express $\displaystyle\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \cos\left(1 + \frac{2k}{n}\right)$ as an *integral.* (Be sure to *include the lower and upper limits* of integration.)

9. Let $f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x \in [0, 2] \setminus \{1\}. \end{cases}$ *Find a partition* P of $[0, 2]$ for which $U(f, P) - L(f, P) < \frac{1}{8}$.

10. True or False: If $f^+ \in R[a,b]$ and if $f^- \in R[a,b]$ then f must be in $R[a,b]$.

11. Give an example of a function $f : [0,1] \to \mathbb{R}$ such that $|f| \in R[0,1]$ but $f \notin R[0,1]$.

12. Let $f = 1_{\mathbb{Q}}$, the *indicator* (or characteristic) function of the set of rational numbers. If P is any partition of $[0,1]$, find the numerical values of $U(f,P)$ and $L(f,P)$. Is $f \in R[0,1]$?

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Prove the following *fixed point theorem*: Suppose $f \in \mathcal{C}[0,1]$ and suppose $0 \le f(x) \le 1$ for all $x \in [0,1]$. Then there exists $c \in [0,1]$ such that $f(c) = c$. (Hint: Consider $g(x) = f(x) - x$.)

B. Suppose $f_n(x) = nxe^{-nx}$ for all $x \in [0,\infty)$.

- (**i**) Find $\|f_n\|_{\text{sup}}$ on the domain $[0,\infty)$. (Hint: You may use the derivative to help you to find this norm.)
- (**ii**) Find the *point-wise limit* $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \in [0,\infty)$.
- (**iii**) Does f_n *converge uniformly* on $[0,\infty)$? Justify your conclusion.

C. Let $p(x) = a_{2n}x^{2n} + \cdots + a_1x + a_0$ be a polynomial of even degree. Prove: If $a_{2n} > 0$, then p has a *minimum value* on $\mathbb{R}$.

Solutions and Class Statistics

1.

a. no

b. yes

c. yes

2. True

3. False.

4.

a. Counterexample: $x \in X_1$ but $0 \cdot x \notin X_1$. More generally, X_n is not closed under either scalar multiplication or vector addition, unless $n = 0$.

b. True: P_n a vector space.

5.

a. 1.

b. ∞

6.

a. Yes

b. No

7.

a. 0

b. Yes, since $\|f_n - 0\|_{\sup} = \frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$.

8. $\displaystyle\int_1^3 \cos x \, dx$

9. For example, we could choose $P = \{0, 0.95, 1.05, 2\}$.

10. True, since $f = f^+ - f^-$ and $R[a,b]$ is a vector space.

11. For example, let $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \cap [0,1] \\ -1 \text{ if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$

12. $U(f, P) = 1$ and $L(f, P) = 0$ for all P. Thus $f \notin R[0, 1]$ because of the Darboux Integrability Criterion.

Remarks about the proofs

Proofs are graded for logical coherence and for logical clarity. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: Be sure to *state which theorems* you are using as reasons for steps in your proof: For example, the Intermediate Value theorem. *Check explicitly that all the hypotheses of that theorem are satisfied* before invoking the conclusion. There are really 3 cases, depending on the values of $g(0)$ and $g(1)$. *Inequalities should be explicit.* The work is not complicated, but it should be shown clearly.

B: Use the derivative to identify the *absolute* maximum of the non-negative functions f_n and thus show that $\|f_n\|_{\sup} = \frac{1}{e}$. Use L'Hospital's Rule to show that the point-wise limit of f_n is identically zero. Since *uniform convergence to a function* f *would imply also point-wise convergence to the same function* f, the fact that $\|f_n - f\|_{\sup} = \frac{1}{e}$, which does not approach zero as $n \to \infty$, implies that the convergence of f_n is *not* uniform.

C: The plan is to factor $p(x) = a_{2n}x^{2n}q(x)$ where the *rational* function $q(x) \to 1$ as $|x| \to \infty$. Make $|x|$ sufficiently big so that $q(x) > \frac{1}{2}$ and then make $|x|$ sufficiently bigger than some $M > 0$ so that so that $a_{2n}x^{2n}q(x) > |a_0| = |p(0)| \geq p(0)$. Then explain why the absolute minimum of the continuous function p on $[-M, M]$ is also an absolute minimum on $\mathbb{R}$. Proofs need to be clear and explicit: *no hand-waving* in advanced calculus. Mathematics isn't a magic show.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4	4	5		
80-89 (B)	6	4	4		
70-79 (C)	2	4	2		
60-69 (D)	1	0	0		
0-59 (F)	2	1	2		
Test Avg	80.9 %	82.8%	83.4%	%	%
HW Avg	8.5	71.8	8.1		
HW/Test Correl	0.76	0.56	0.66		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{13}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.