Name: ________________________________________

Instructions. Show all work in the space provided. Indicate clearly if you continue on the back side, and write your name at the top of the scratch sheet if you will turn it in for grading. No books or notes are allowed. A scientific calculator is ok - but not needed. The maximum total score is 100.

Part I - 48 points. Answer 8 of the following 12 questions. Circle the numbers of the 8 questions you want counted - no more than 8! Detailed explanations are not required, but they may help with partial credit and are risk free!

1. Let $\epsilon > 0$. How large must $n \in \mathbb{N}$ be in order to insure that $\frac{1}{n^2} < \epsilon$?

2. True or False: $||a| - |b|| \geq |a - b|$.

In questions #3 through #5, determine whether or not the sequence is Cauchy.

3. $x_n = \frac{(-1)^n}{n}$

4. $x_n = \frac{n^3+1}{n^2}$

5. Let $x_n$ follow this pattern: $0, 1, \frac{1}{2}, 0, \frac{1}{3}, 1, \frac{3}{4}, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{2}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, 1, \ldots$
6. True or false: If the sequence $x_n \to \infty$ then $x_n$ is monotone increasing.

7. Let $x_n = (-1)^n a + \frac{1}{n}$, where $a > 0$ is a constant. Find $\limsup x_n$ and $\liminf x_n$.

8. True or False: If $\{x_n | n \in \mathbb{N}\}$ has no upper bound, then $\sup\{x_n, x_{n+1}, \ldots\} = \infty$ for all $n$.

9. Give an example of an unbounded sequence $x_n$ which has a convergent subsequence.

10. Give an example of a divergent sequence $x_n$ and a divergent sequence $y_n$ such that $x_n - y_n$ converges.

11. Give an example of a decreasing nest of non-empty open finite intervals $(a_1, b_1) \supseteq (a_2, b_2) \supseteq \cdots$ such that $\cap_{k=1}^{\infty} (a_k, b_k) = \emptyset$. 
12. True or False: The set \( E = \left\{ \frac{\sqrt{2}}{n} \mid n \in \mathbb{N} \right\} \) is a countable set.

Part II - 52 points. Prove carefully two of the following three theorems. Circle the letters of the two proofs to be counted - *no more than two!* You may write the proofs below, on the back, or on scratch paper.

A. Suppose \( A \) and \( B \) are subsets of \( \mathbb{R} \), both *non-empty*, with the special property that \( a \leq b \) for all \( a \in A \) and for all \( b \in B \). Prove: \( \sup(A) \leq \inf(B) \). (Hint: Every \( b \) is an upper bound of \( A \). So how does \( \sup(A) \) relate to each \( b \in B \)?)

B. If \( x_n + y_n \) converges and if \( x_n - y_n \) converges, prove that \( x_n \) converges and \( y_n \) converges.

C. Let \( E \subseteq \mathbb{R} \) be any *unbounded* set. Find an open covering of \( E \) which has no finite sub-covering. Prove that you have chosen an open covering and that it has no finite sub-covering.