Print Your Name Here: ____________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Suppose $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{n} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \ n \in \mathbb{N}. \end{cases}$ True or False: $f \in \mathcal{R}[0,1]$.

2. True or False: the function $f_n(t) = \dfrac{1}{t \ln n} \to 0$ uniformly on $[1, \infty)$.

3. True or False: $\displaystyle\lim_{n\to\infty} \int_1^n \frac{1}{t \ln n}\, dt = \int_1^\infty \lim_{n\to\infty} \frac{1}{t \ln n}\, dt.$

4. *Apply the Cauchy-Schwarz inequality* to fill in the right side of the following inequality: $\displaystyle\int_0^\pi \sqrt{x}\sqrt{\sin x}\, dx \le$ ____.

5. Let $g \in \mathcal{C}[a,b]$, a normed vector space equipped with the sup-norm. Define $T_g : \mathcal{C}[a,b] \to \mathbb{R}$ by $T_g(f) = \displaystyle\int_a^b f(x)g(x)\, dx$ for all $f \in \mathcal{C}[a,b]$. Find a *bound* K for the linear functional T_g in terms of the function g.

6. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ Calculate $f'(0)$ from the definition of the derivative.

7. Let $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$ True or False: $f'(0)$ exists.

8. True or Give a Counterexample: If f is a *strictly increasing* differentiable function on (a, b) then $f'(x) > 0$ for all $x \in (a, b)$.

9. True or Give a Counterexample: If $f \in \mathcal{R}[a, b]$ then the function $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$.

10. True or Give a Counterexample: If $f \in \mathcal{R}[a, b]$ then the function $F(x) = \int_a^x f(t)\,dt$ then $F'(x)$ exists and equals $f(x)$ for all $x \in [a, b]$.

11. True or Give a Counterexample: If $F'(x) = f(x)$ for all $x \in [a, b]$ then f is Riemann integrable on $[a, b]$ and $\int_a^b f(x)\,dx = F(b) - F(a)$.

12. Find $\displaystyle\int_0^1 g(x)\,dx$ if $g(x) = \begin{cases} 2x \cos \frac{\pi}{x} + \pi \sin \frac{\pi}{x} & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. If $f \in \mathcal{R}[a, b]$, prove: $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. (Hint: Write the left side by expressing $f = f^+ - f^-$ and use the fact that f^+ and f^- are both nonnegative functions. This is a triangle inequality for integrals.)

B. If f and g are in $\mathcal{R}[a, b]$, we say f is *orthogonal* to g, denoted by $f \perp g$, if and only if $\langle f, g\rangle = 0$. Prove that $f \perp g \Leftrightarrow \|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$. (This is a modern analogue of the *Pythagorean Theorem in infinite dimensional space.*)

C. Let f be *uniformly continuous* on the finite open interval (a, b). Prove that $\displaystyle\lim_{x\to a+} f(x)$ exists. (Hint: Consider first any sequence $x_n \to a+$ and prove that $f(x_n)$ is Cauchy. But remember that one sequence is not sufficient to invoke the sequential criterion for limits of functions.)

Solutions and Class Statistics

1. True: the function f is monotone.

2. True: $\left\|\dfrac{1}{t\ln n}-0\right\|_{\text{sup}}=\dfrac{1}{n}\to 0$ as $n\to\infty$.

3. False: The left side is 1 but the right side is 0. Compare with the preceding problem.

4. $\displaystyle\int_0^{\pi}\sqrt{x}\sqrt{\sin x}\,dx\le\pi$.

5. $K=\displaystyle\int_a^b|g(x)|\,dx$ is the best choice.

6. $f'(0)=0$.

7. True: $f'(0)=0$.

8. Counterexample: $f(x)=x^3$ is strictly increasing on $(-1,1)$ but $f'(0)=0$.

9. True.

10. Counterexample: let f be any function lacking the intermediate value property, so that it cannot be the derivative of another function. For example, you could use a step function such as $f(x)=1_{[0,1]}$ on the interval $[0,2]$.

11. Counterexample: let $F(x)=\begin{cases}x^2\sin\frac{1}{x^2} & \text{if } 0<|x|\le 1,\\ 0 & \text{if } x=0.\end{cases}$ Then $F'(x)$ exists on [-1,1] but $F'=f\notin\mathcal{R}[-1,1]$ because it is not bounded near $x=0$.

12. $\displaystyle\int_0^1 g(x)\;dx=x^2\cos\left(\frac{\pi}{x}\right)\Big|_0^1=-1$. Note that the antiderivative $G(x)$ must have the value 0 at 0 in order to be differentiable there.

Remarks about proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, please bring me your test and also the graded homework from which the questions in Part II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Do take care not to claim false inequalities, and not to assume what you are asked to prove.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4				
80-89 (B)	0				
70-79 (C)	4				
60-69 (D)	2				
0-59 (F)	0				
Test Avg	80.5%	%	%	%	%
HW Avg	6.8				
HW/Test Correl	0.73				

Correlation coefficients are between 1 and -1 always, being the cosine of the angle between two data vectors. Coefficients equal to or greater than 0.6 are considered strongly positive in statistics.