

Print Your Name Here: _____

Show all work in the space provided and *keep your eyes on your own paper*. Indicate clearly if you continue on the back. Write your **name** at the **top** of the *scratch sheet if you will hand it in to be graded*. **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8!* Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: If a function f defined on $[0, 1]$ has no antiderivative, then $f \notin \mathcal{R}[0, 1]$.
2. Let $T : \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ be defined by $T(f) = \int_0^1 f(x) (1 + x^2) dx$. Find a constant K for which $|T(f)| \leq K \|f\|_{\text{sup}}$ for all $f \in \mathcal{C}[0, 1]$.
3. What inequality would you use to prove that $\int_0^\pi \sqrt{x \sin x} dx \leq \pi$? (Either state the name of the inequality or write the inequality as a formula if you don't recall the name.)
4. True or False: If $f \in \mathcal{R}[0, 1]$, then $\int_0^1 xf(x) dx \leq \frac{1}{\sqrt{3}} \left[\int_0^1 [f(x)]^2 dx \right]^{\frac{1}{2}}$.

5. Give an example of $f \in \mathcal{R}[0, 1]$ such that $\|f\|_2 = 0$ yet $f(x)$ is not identically 0 on $[0, 1]$.
6. True or give a counterexample: If $\delta > 0$ and if $f \in \mathcal{R}[a, b]$ is such that $|f(x)| \geq \delta$ for all $x \in [a, b]$, then $\frac{1}{f} \in \mathcal{R}[a, b]$.
7. True or give a counterexample: If f is uniformly continuous and differentiable on (a, b) , then $f'(x)$ is bounded on (a, b) .
8. True or give a counterexample: If $|f'(x)| \leq 100$ for all $x \in \mathbb{R}$ then f is uniformly continuous on \mathbb{R} .
9. True or give a counterexample: If f is differentiable on $[a, b]$ and if f increases strictly on $[a, b]$ then $f'(x) > 0$ for all $x \in [a, b]$.
10. True or give a counterexample: Every *uniformly* continuous function on the finite open interval (a, b) is the restriction to (a, b) of a continuous function on $[a, b]$.

11. Give an example of a function $F(x)$ for which $F'(x)$ exists for all $x \in [-1, 1]$ yet F' is not bounded on $[-1, 1]$.

12. Let $f(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$ Find $\int_{-2}^2 f(x) dx$.

Part II: Proofs. Prove carefully **2** of the following 3 theorems for 26 points each. **Circle** the letters of the 2 proofs to be counted in the list below—no more than 2! You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Suppose $f \in \mathcal{C}(a, b)$ and also that there is $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Prove: $f \in \mathcal{R}[a, b]$. (Hint: By the Variant form of the Darboux Integrability Criterion, it suffices to show that if $\epsilon > 0$ there exists a partition \mathcal{P} of $[a, b]$ for which $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.)

B. If $f, g \in \mathcal{R}[a, b]$, we say f is *orthogonal* to g , denoted by $f \perp g$, if and only if $\langle f, g \rangle = \int_a^b f(x)g(x) dx = 0$. Prove that $f \perp g \Leftrightarrow \|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$. (This is a modern analogue of the *Pythagorean Theorem*.)

C. Suppose f' and g' exist on $[a, b]$ and suppose $f', g' \in \mathcal{R}[a, b]$. Prove:

(i) $(fg)'$, $f'g$, and fg' are all the Riemann integrable functions on $[a, b]$.

(ii) $\int_a^b f(x)g'(x) dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x) dx$. (Hint: Consider $\int_a^b (fg)'(x) dx$.)

Solutions and Class Statistics

1. False: every step function is Riemann integrable on $[0, 1]$ but it has no antiderivative because it lacks the intermediate value property.
2. $K = \frac{4}{3}$
3. The Cauchy-Schwarz inequality, or you could state the inequality if you do not recall its name. See exercise 3.48(a).
4. True: see exercise 3.50
5. For example, let f be the indicator function of a single point, such as $x = \frac{1}{2}$.
6. True: see exercise 3.47.
7. Counterexample: let $f(x) = \sqrt{|x|}$ on $(0, 1)$.
8. True: see exercise 4.20a.
9. Counterexample: let $f(x) = x^3$ on $[-1, 1]$.
10. True, by Exercise 4.20(b).
11. For example, let $F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < |x| \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$
12. $\int_{-2}^2 f(x) dx = 8 \sin\left(\frac{1}{2}\right)$.

Remarks about the proofs

Proofs are graded for logical coherence. Be sure to state what is your hypothesis (the assumption) and what conclusion you are seeking to prove. Then include justifications for each step. Your job is to show me through your writing that you understand the reasoning. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: No one chose part (A). This was a homework problem: 3.26.

B: Remember that $\langle f, g \rangle$ can be positive, negative, or zero. Also, since this is a necessary and sufficient condition for $\|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$ the proof must be bidirectional, not just one way or the other.

C: One needs to know the derivative of a product of two differentiable functions. Also, one needs to know that differentiability implies continuity, which implies Riemann integrability. One uses the fact that $R[a, b]$ is a vector space and that it is also closed under pointwise multiplication of functions.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	5				
80-89 (B)	3				
70-79 (C)	1				
60-69 (D)	0				
0-59 (F)	0				
Test Avg	89.6%	%	%	%	%
HW Avg	86.8%				
HW/Test Correl	0.45				

The Correlation Coefficient is the cosine of the angle between two data vectors in \mathbb{R}^9 —one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a positive correlation with performance on the homework. (In my experience, this statistic is somewhat unstable in classes with low enrollment.)