Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Give an example of a sequence f_n for which $f_n' \to 0$ uniformly on $\mathbb{R}$, yet $f_n(x)$ *diverges* for all $x \in \mathbb{R}$.

2. Find $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^{100}}{x^{0.1}}$.

3. If P is any polynomial, find $\displaystyle\lim_{h\to 0} \frac{P(x+2h)+P(x-2h)-2P(x)}{h^2}$

4. Find a formula for a_k so that $s_n = \sum_{k=1}^{n} a_k = \log(n+1)$, for all n.

5. Give an example of $x_k \to 0$ for which $\sum_{k=1}^{\infty}(-1)^{k+1}x_k$ *diverges.*

6. If $p(x) = x^6 - 5x^4 - 3x^3 + 1$ and if $p(x) = \displaystyle\sum_{k=0}^{6} \frac{p^{(k)}(1)}{k!}(x-1)^k + R_6(x)$, find the numerical value of the Taylor remainder $R_6(x)$.

7. Test for Convergence or divergence: $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$.

8. Test for Convergence or divergence: $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$.

9. Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ converge absolutely, converge conditionally, or diverge?

10. List the three terms $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$, $\displaystyle\int_1^n \frac{1}{x}\,dx$, and $1 + \displaystyle\int_1^n \frac{1}{x}\,dx$ correctly in the form $a \le b \le c$.

11. Does the series $\displaystyle\sum_{0}^{\infty} \left(\frac{-3}{\pi}\right)^k$ converge absolutely, converge conditionally, or diverge? If it converges, what is the numerical value of the sum?

12. Find the numerical value of $\displaystyle\sum_{1}^{\infty} \frac{2}{k(k+2)}$. (Hint: telescoping series.)

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $f_n(x) = \sin^n x$ for all $x \in [0, \pi]$. Prove that f_n' is *not* uniformly convergent on $[0, \pi]$. (Hint: Suppose false and apply a theorem about uniform convergence and derivatives to deduce a contradiction. Be sure to check all the hypotheses of that theorem.)

B. A function $f : \mathbb{R} \to \mathbb{R}$ is called a *contraction* of $\mathbb{R}$ if and only if there exists a constant $r \in [0, 1)$ such that for all x and x' in $\mathbb{R}$ we have $|f(x) - f(x')| \le r|x - x'|$. Let f be a contraction of $\mathbb{R}$, with corresponding constant r.

- **(i)** Let $x_0 \in \mathbb{R}$ be arbitrary and define a sequence x_n by $x_n = f(x_{n-1})$, for each $n \in \mathbb{N}$. Show that $|x_{n+1} - x_n| \le r^n |x_1 - x_0|$.
- **(ii)** Prove that the sequence x_n in part (i) is a Cauchy sequence.

C. Let $S = \limsup \frac{x_{k+1}}{x_k}$, where $x_k > 0$ for all $k \in \mathbb{N}$. Prove: If $S < 1$, then the positive-term series $\displaystyle\sum_{k=1}^{\infty} x_k$ converges. Take care that your proof takes account of what lim sup means.

Solutions and Class Statistics

1. For example, let $f_n(x) = n$ for all $x \in \mathbb{R},\ n \in \mathbb{N}$.

2. $\lim_{x\to\infty} \frac{(\ln x)^{100}}{x^{0.1}} = 0$ applying L'Hospital's Rule 100 times.

3. $\lim_{h\to 0} \frac{P(x+2h)+P(x-2h)-2P(x)}{h^2} = 4P''(x)$.

4. $a_k = S_k - S_{k-1} = \log\left(1+\frac{1}{k}\right)$, where we assume $S_0 = 0$, the sum of no terms.

5. For example, let $x_k = \frac{(-1)^{k+1}}{k}$.

6. $R_6(x) = 0$

7. The limit of the ratios in the ratio test is $e > 1$: Hence the series diverges. Or one can use the nth term test.

8. There was a typo on the original test papers: k should range from 2 to ∞ since $\log 1 = 0$. Using the n^{th} root test, the limit of the n^{th} roots is $0 < 1$: Hence the series converges. The grading was lenient because of the typo.

9. Converges absolutely by the p-series test.

10. $\int_1^n \frac{1}{x}\,dx \le \sum_{k=1}^{n} \frac{1}{k} \le 1 + \int_1^n \frac{1}{x}\,dx$, from the proof of the integral test.

11. $\sum_0^\infty \left(\frac{-3}{\pi}\right)^k = \frac{\pi}{\pi+3}$ with the convergence being absolute, by the geometric series test.

12. $\sum_1^\infty \frac{2}{k(k+2)} = \frac{3}{2}$

Remarks about proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, please bring me your test and also the graded homework from which the questions in Part II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Do take care not to claim false inequalities, and not to assume what you are asked to prove.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4	3			
80-89 (B)	0	2			
70-79 (C)	4	1			
60-69 (D)	2	0			
0-59 (F)	0	4			
Test Avg	80.5%	72.5%	%	%	%
HW Avg	6.8	6.7			
HW/Test Correl	0.73	0.72			

Correlation coefficients are between 1 and -1 always, being the cosine of the angle between two data vectors. Coefficients equal to or greater than 0.6 are considered strongly positive in statistics.