Print Your Name Here: ____________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: There exists a rearrangement of $\sum_1^\infty (-1)^{n+1}\frac{1}{n^2}$ that converges to $\pi\sqrt{2}$.

2. If $\sum_1^\infty |x_k| < \infty$, what do you know about $\sum_1^\infty x_k^+$ and about $\sum_1^\infty x_k^-$.

3. Let e_j and f_k be any two sequences, $j, k = 0, 1, 2, \ldots$. Let c be the *Cauchy Product* of e with f. Write out the lth term c_l in terms of the appropriate terms of the form e_j and f_k.

4. Evaluate the double sum $\sum_{j,k=0}^{\infty} \frac{1}{2^j 3^k}$.

5. Give an example of a double sum $\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} a_{j,k}$ in which each *column sum* $c_k = 0$, making the sequence c_k absolutely summable, and yet $\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} |a_{j,k}|$ diverges.

6. If $x_k = \frac{(-1)^k}{2^k}$ for each $k \in \mathbb{N}$, find $\|x\|_1$.

7. True or False: Suppose $T_n \in V'$ for all $n \in \mathbb{N}$ and suppose for each individual $v \in V$ the sequence $T_n(v)$ converges to some limit, which we define to be $T(v)$. Then T is linear.

8. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_1$ by letting $x_k^{(n)} = \begin{cases} 1 & \text{if } k \leq n, \\ \frac{1}{k^2} & \text{if } k > n \end{cases}$. Define a sequence x by letting $x_k = \lim_{n\to\infty} x_k^{(n)}$, for all $k \in \mathbb{N}$. Find $\|x\|_1$.

9. Let $f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$ on $\mathbb{R}$. True or False: $f \in \mathcal{C}^1(\mathbb{R})$.

10. True or False: $\sum_{k=1}^{\infty} \sin^k x$ converges uniformly on $\left[0, \frac{\pi}{2}\right)$.

11. Find $\sum_{k=1}^{\infty} \frac{1}{k2^k}$. (Hint: Begin with the series $\sum_{k=0}^{\infty} t^k$ and integrate from 0 to 1/2.)

12. Let $f_k(x) = \frac{(-1)^{k+1}}{k} x^k$. True or False: $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on [-1,1].

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. If c is a summable sequence and if $x \in [0,1]$, prove that $\sum_{k=0}^{\infty} c_k x^k$ converges.

B. Let V be a normed linear space. If $T \in V'$, prove $|T(v)| \leq \|T\| \cdot \|v\|$ for all $v \in V$

C. Let $f_k(x) = \frac{(-1)^{k+1}}{k} x^k$.

- **(i)** Prove: $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on [0,1]. (Hint: Use the error estimate from the Alternating Series Test.)
- **(ii)** Find $\|f_k\|_{\sup}$ on [0,1]. Can the Weierstrass M-test be used to prove the uniform convergence of $\sum_{k=1}^{\infty} f_k(x)$ on [0,1]? Why or why not?

Solutions and Class Statistics

1. False: the given series is absolutely convergent.

2. $\sum_1^\infty x_k^+ < \infty$ and about $\sum_1^\infty x_k^- < \infty$.

3. $c_l = \sum_{j=0}^{l} e_j f_{l-j}$.

4. This is the product of two absolutely convergent infinite series. First evalute the column sums and then evaluate the sum of these, finding the result 3.

5. For example, for each k, let the kth column sequence a_{jk} be the sequence $1, -1, 0, 0, \underline{0}, \ldots$.

6. 1

7. True.

8. $\|x\|_1 = \infty$

9. True, because the sum of the term by term derivatives converges uniformly by the M-test, and the original series converges at $x = 0$ for example.

10. False.

11. $\ln 2$.

12. False: the series diverges at $x = -1$.

Remarks about proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, please bring me your test and also the graded homework from which the questions in Part II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Do take care not to claim false inequalities, and not to assume what you are asked to prove.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4	3	3		
80-89 (B)	0	2	4		
70-79 (C)	4	1	3		
60-69 (D)	2	0	0		
0-59 (F)	0	4	0		
Test Avg	80.5%	72.5%	84.7%	%	%
HW Avg	6.8	6.7	6.5		
HW/Test Correl	0.73	0.72	0.73		

Correlation coefficients are between 1 and -1 always, being the cosine of the angle between two data vectors. Coefficients equal to or greater than 0.6 are considered strongly positive in statistics.