Print Your Name Here: ____________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Suppose a sequence a_k is summable and a sequence b_k is absolutely summable. True or Give a Counterexample: $\sum_{k=1}^{\infty} a_k b_k$ is absolutely convergent.

2. Find $\frac{1}{3} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots + \frac{1}{3^k} - \frac{1}{4^k} + - \cdots$.

3. True or False: The alternating harmonic series can be rearranged to converge to π^e.

4. True or Give a Counterexample: If a sequence x is *conditionally* summable and a sequence y is not identically 0, then the countable family of terms $x_j y_k$ cannot be absolutely summable in any order.

5. Give an example of a rectangular array $a_{j,k}$ of real numbers for which each *column sum* $c_k = \sum_{j=1}^{\infty} a_{j,k} = 0$, making the sequence c_k absolutely summable, and yet $\sum_{(j,k)\in\mathbb{N}^2} |a_{j,k}| = \infty$.

6. True or Give a Counterexample: If c is a *bounded* sequence and if $x \in [-1,1]$, then $\sum_{k=0}^{\infty} c_k x^k$ converges.

7. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_1$ by letting $x_k^{(n)} = \frac{n+1}{n2^k}$, for all $k \in \mathbb{N}$. Find $\left\|x^{(n)}\right\|_1$.

8. If $x_k^{(n)} = \frac{n+1}{n2^k}$, for all $k \in \mathbb{N}$ and if $x_k = \lim_{n\to\infty} x_k^{(n)}$, *find* $\left\|x^{(n)} - x\right\|_1$.

9. Let $f_k(x) = \frac{(-1)^{k+1}}{k} x^{2k-1}$. True or False: $\sum_{k=1}^{\infty} f_k(x)$ converges *uniformly* on [0,1].

10. Does $\sum_{k=1}^{\infty} e^{-kx}$ converge *uniformly* on $[1,\infty)$?

11. Does $\sum_{k=1}^{\infty} \sin^k x$ converge *uniformly* on $[0, \delta]$, where $0 < \delta < \frac{\pi}{2}$?

12. Does $\sum_{k=1}^{\infty} \sin^k x$ converge *uniformly* on $\left[0, \frac{\pi}{2}\right)$?

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. If $\sum_{k=1}^{\infty} x_k$ is *conditionally* convergent, prove that $\sum_{k=1}^{\infty} x_k^+ = \infty = \sum_{k=1}^{\infty} x_k^-$. Here x_k^+ and x_k^- are defined as in the proof of Dirichlet's Theorem. (*Be sure to explain sums of infinite series in terms of limits of partial sums.*)

B. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_1$ by letting $x_k^{(n)} = \begin{cases} 1 & \text{if } k \leq n, \\ \frac{1}{k^2} & \text{if } k > n. \end{cases}$

- **(i)** Show that $x^{(n)} \in l_1$ by showing $\left\| x^{(n)} \right\|_1 < \infty$.
- **(ii)** Define a sequence x by letting $x_k = \lim_{n\to\infty} x_k^{(n)}$, for all $k \in \mathbb{N}$. Is $x \in l_1$? Prove your answer, yes or no.
- **(iii)** Is $x^{(n)}$ a Cauchy sequence in l_1? Prove your answer, yes or no.

C. Let $f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$ on $\mathbb{R}$. Prove that $f \in \mathcal{C}^1(\mathbb{R})$ and find an expression for $f'(x)$ in terms of an infinite series. Justify all your conclusions.

Solutions and Class Statistics

1. True, since the sequence a_k must be bounded.

2. $\frac{1}{6}$. See Exercise 5.20.

3. True

4. True: see exercise 5.25.

5. For example, let $a_{1,k} = 1,\ a_{2,k} = -1$ for all k, and $a_{j,k} = 0$ for all $j > 2$.

6. Counterexample: Let $c_k = (-1)^k$ and $x = -1$. Compare with exercise 5.21.

7. $\left\|x^{(n)}\right\|_1 = \dfrac{n+1}{n}$. See exercise 5.36.

8. $\left\|x^{(n)} - x\right\|_1 = \frac{1}{n}$. See exercise 5.36.

9. True, by the error estimate in the Alternating Series Test. See Exercise 5.39.

10. Yes, by the Weierstrass M-Test. See Exercise 5.43.

11. Yes, by the M-test.

12. No. If this series did converge uniformly, its partial sums would be bounded in the sup-norm, but they are not.

Remarks about the proofs

Proofs are graded for logical coherence. Be sure to state what is your hypothesis (the assumption) and what conclusion you are seeking to prove. Then include justifications for each step. Your job is to show me through your writing that you understand the reasoning. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: The proofs offered were pretty good, but I have suggestions for several of you to be more careful with language. Note that $\infty \notin \mathbb{R}$, so ∞ does not participate in the arithmetic of the real number system. That is why we need to be very careful to work with limits of *partial* sums and not to write meaningless statements such as "$\infty \pm Q$." Also a statement such as $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$ can be false and thus requires some justification. For example, $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k}\right)$ is not equal to

$\infty - \infty$ since the latter expression is meaningless. This conditionally convergent sum is actually a real number between $\frac{1}{2}$ and 1.

B: Part (i) is a matter of showing that for each fixed n, the sequence $x_k^{(n)}$ is (absolutely) summable. For (ii) it suffices to explain why $x_k = 1$ for all k and this sequence is not (absolutely) summable. For (iii) it is best show that $x^{(n)}$ is not Cauchy since l_1 is *complete (that is, a Banach space)*, so that the limit of a Cauchy sequence in l_1 must be in l_1.

C: Use the Weierstrass M-test to show that $\sum f_k$ and $\sum f_k'$ are both uniformly convergent on $\mathbb{R}$. Differentiability of $f = \sum f_k$ and the equality between f' and $\sum f_k'$ requires careful checking of all the hypotheses of Theorem 5.5.1 part (iii). Take care with this because the theorem works on finite intervals, but $\mathbb{R}$ is infinite. So you'll have to prove differentiability and equality with the sum of the term-by-term derivatives at each arbitrary $x \in \mathbb{R}$.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	5	2	5		
80-89 (B)	3	3	2		
70-79 (C)	1	1	2		
60-69 (D)	0	3	0		
0-59 (F)	0	0	0		
Test Avg	89.6%	79.4%	88.3%	%	%
HW Avg	86.8%	87.6%	82.1%		
HW/Test Correl	0.45	0.63	0.68		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^9$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework. (In my experience, this statistic is somewhat unstable in classes with low enrollment.)