Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 200.

Part I: Short Questions. Answer **12** of the 18 short questions: 8 points each. **Circle** the **numbers** of the 12 questions that you want counted—*no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points.

1. True or False: $\lim_{n\to\infty} \int_1^n \frac{1}{t\ln n}\,dt = \int_1^\infty \lim_{n\to\infty} \frac{1}{t\ln n}\,dt.$

2. *Apply the Cauchy-Schwarz inequality* to fill in the right side of the following inequality: $\int_0^\pi \sqrt{x}\sqrt{\sin x}\,dx \le$ ____.

3. True or give a counterexample: The function $|f| \in \mathcal{R}[a,b] \Leftrightarrow f \in \mathcal{R}[a,b]$.

4. Let $f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x \le 1 \end{cases}$ for each $n > 1$. Find

a. $\lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$

b. $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

5. True or give a counter-example: If $f \in \mathcal{R}[a,b]$ then the function $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$.

6. Find $\int_0^1 g(x)\,dx$ if $g(x) = \begin{cases} 2x\cos\frac{\pi}{x} + \pi\sin\frac{\pi}{x} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$

7. Give an example of a sequence f_n for which $f_n' \to 0$ uniformly on $\mathbb{R}$, yet $f_n(x)$ *diverges* for all $x \in \mathbb{R}$.

8. Let $f_n(x) = \dfrac{1}{n}\sin\left(n^2 x\right)$. True or False: $f_n \to 0$ uniformly on $\mathbb{R}$ and $f_n'(0) \to 0$.

9. Give an example of $x_k \to 0$ for which $\sum_{k=1}^{\infty}(-1)^{k+1}x_k$ *diverges.*

10. Test for *absolute convergence, conditional convergence, or divergence:* $\displaystyle\sum_{k=1}^{\infty} a_k$ where

$a_k = \begin{cases} \frac{1}{j} & \text{if } k = 2j-1, \\ \frac{-1}{j} & \text{if } k = 2j. \end{cases}$

11. Test for convergence or divergence: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

12. Give an example of *two* sequences, x_k *and* y_k, such that $\dfrac{x_{k+1}}{x_k} \to 1$ and $\dfrac{y_{k+1}}{y_k} \to 1$, but $\displaystyle\sum_{k=1}^{\infty} x_k$ converges whereas $\displaystyle\sum_{k=1}^{\infty} y_k$ diverges.

13. If $x_k = \dfrac{(-1)^k}{2^k}$ for each $k \in \mathbb{N}$, find $\|x\|_1$.

14. If c_k is a bounded sequence, for which values of x is $\displaystyle\sum_{k=1}^{\infty} c_k x^k$ absolutely convergent?

15. True or give a counter-example: If x is a conditionally summable sequence and y is a sequence that is not identically 0, then the countable family $\{x_j y_k \mid j, k \in \mathbb{N}\}$ cannot be absolutely summable in any order.

16. If $y_k = \dfrac{1}{k}$ for all $k \in \mathbb{N}$ and $T_y(x) = \sum_{k=1}^{\infty} y_k x_k$ for all $x \in l_1$, find $\|T_y\|$.

17. Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ for all $x \in \mathbb{R}$. True or False: $f \in \mathcal{C}(\mathbb{R})$.

18. True or False: the series $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges uniformly on $(-1, 1)$.

Part II: Proofs. *Prove carefully* **4** of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 4!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. If $f \in \mathcal{R}[a, b]$, prove: $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. (Hint: Write the left side by expressing $f = f^+ - f^-$ and use the fact that f^+ and f^- are both nonnegative functions. This is a *triangle inequality* for integrals.)

B. Let $F(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } 0 < |x| \le 1, \\ 0 & \text{if } x = 0. \end{cases}$

- **(i)** (8) Prove that $F'(x)$ exists for *all* $x \in [-1, 1]$, *including* $x = 0$, and let $f(x) = F'(x)$.
- **(ii)** (8) Show that $f \in R[-1, 1]$.
- **(iii)** (10) Find $\int_{-1}^1 f(x)\,dx$. (See Fig. 1.)

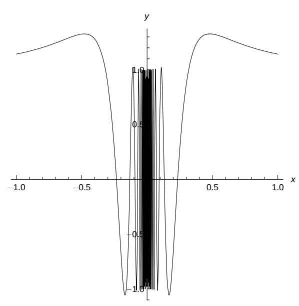

Figure 1: $2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right),\ x \in [-1, 1] \setminus \{0\}$.

C. Prove that e is irrational. (Hint: Suppose false, so that $e = \frac{p}{q}$, where $p, q \in \mathbb{N}$. Write $e = e^1 = P_n(1) + R_n(1)$, where P_n is the nth Taylor polynomial for e^x and R_n is the nth Taylor Remainder. Multiply both sides by $n!$, and deduce a contradiction when $n \in \mathbb{N}$ is sufficiently large.)

D. Suppose $x_k \ge 0$ for all $k \in \mathbb{N}$, and suppose $\sqrt[k]{x_k} \to L$ as $k \to \infty$. If $L < 1$, prove that $\sum_{k=1}^\infty x_k$ converges. (This is part of the nth root test.)

E. For all $n \in \mathbb{N}$ define a sequence $x^{(n)} \in l_1$ by letting $x_k^{(n)} = \begin{cases} 1 & \text{if } k \le n, \\ \frac{1}{k^2} & \text{if } k > n. \end{cases}$

- **(i)** (8) Show that $x^{(n)} \in l_1$ by showing $\left\|x^{(n)}\right\|_1 < \infty$.
- **(ii)** (8) Define a sequence x by letting $x_k = \lim_{n\to\infty} x_k^{(n)}$, for all $k \in \mathbb{N}$. Is $x \in l_1$? Prove your answer, yes or no.
- **(iii)** (10) Is $x^{(n)}$ a Cauchy sequence in the l_1-norm, $\|\cdot\|_1$? Prove your answer, yes or no.

F. $\frac{1}{t} = \sum_{k=0}^\infty (-1)^k (t-1)^k$ provided that $|t - 1| < 1$.

- **(i)** (10) Integrating from $t = 1$ to $t = x$, for all $x \in (0, 2)$, find a power series expansion in powers of $(x - 1)$ for $\ln x$.
- **(ii)** (16)Prove that the power series representation of $\ln x$ found in (a) remains valid for $x = 2$. What about $x = 0$? Explain your conclusions.

Solutions and Class Statistics

1. False: The left side is 1 but the right side is 0.

2. $\displaystyle\int_0^{\pi} \sqrt{x}\sqrt{\sin x}\,dx \le \pi.$

3. Counter-example: Let $f(x) = 1_{\mathbb{Q}\cap[0,1]} - 1_{[0,1]\setminus\mathbb{Q}}$

4.

a. $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$

b. $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{2}$.

5. True, since f is bounded.

6. -1

7. For example, let $f_n(x) = n$ for all $x \in \mathbb{R},\ n \in \mathbb{N}$.

8. False: $f_n'(0)$ diverges as $n \to \infty$.

9. For example, let $x_k = \frac{(-1)^{k+1}}{k}$.

10. conditionally convergent

11. convergent, by the nth root test.

12. For example, let $x_k = \dfrac{1}{k^2},\ y_k = \dfrac{1}{k}$.

13. 1

14. $|x| < 1$.

15. True

16. $\|T_y\| = 1$, the sup-norm of the sequence y.

17. True, since this series of continuous functions is uniformly convergent by the Weierstrass M-test.

18. False.

Remarks about proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, please bring me your test and also the graded homework from which the questions in Part II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Do take care not to claim false inequalities, and not to assume what you are asked to prove.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	4	3	3	4	4
80-89 (B)	0	2	4	3	3
70-79 (C)	4	1	3	3	3
60-69 (D)	2	0	0	0	0
$\rightleftharpoons 0-59(F)$	0	4	0	0	0
Test Avg	80.5%	72.5%	84.7%	86.2%	87.6%
HW Avg	6.8	6.7	6.5	6.4	6.4
HW/Test Correl	0.73	0.72	0.73	0.68	0.68

Correlation coefficients are between 1 and -1 always, being the cosine of the angle between two data vectors. Coefficients equal to or greater than 0.6 are considered strongly positive in statistics.