1. Express \( \lim_{n \to \infty} \frac{2}{n} \sum_{k=1}^{n} \cos(1 + \frac{2k}{n}) \) as an integral. (Be sure to include the lower and upper limits of integration.)

2. Let \( f(x) = \begin{cases} 1, & \text{if } x \in \{ \frac{1}{n} | n \in \mathbb{N} \} \\ 0, & \text{if } x \in [0, 1] \setminus \{ \frac{1}{n} | n \in \mathbb{N} \} \end{cases} \). Either find \( \int_{0}^{1} f(x) \, dx \) if \( f \in \mathbb{R}[0, 1] \) or state that \( f \not\in \mathbb{R}[0, 1] \).

3. Let \( f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{\sqrt{x}}, & \text{if } 0 < x \leq 1 \end{cases} \). Either find \( \int_{0}^{1} f(x) \, dx \) if \( f \in \mathbb{R}[0, 1] \), or else state that \( f \not\in \mathbb{R}[0, 1] \).
4. Give an example of $f$ such that $|f| \in \mathbb{R}[a,b]$ yet $f \notin \mathbb{R}[a,b]$.

5. Let $f(x) = \begin{cases} \sin \frac{\pi}{x}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$. Find $\int_a^b f - \int_a^b f$.

6. Answer True or False for each separate part below.

(a) If $f \in \mathbb{R}[a,b]$ then $f \in C[a,b]$.

(b) If $f$ is monotone decreasing on $[a,b]$, then $f \in \mathbb{R}[a,b]$.

7. Let $f_n(x) = \begin{cases} n, & \text{if } 0 < x \leq \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < x \leq 1 \end{cases}$ for all $n \in \mathbb{N}$. For each $x \in [0,1]$, find $f(x) = \lim_{n \to \infty} f_n(x)$, the point-wise limit.

8. Let $T : C[0,3] \to \mathbb{R}$ be defined by $T(f) = f(1) - f(2)$. True or False: $T$ is a bounded linear functional on $C[0,3]$, equipped with the sup-norm.
9. \( T : C[0, 1] \to \mathbb{R} \) by \( T(f) = \int_0^1 (1 + x^2) f(x) \, dx \), for all \( f \in C[0, 1] \). True or False: \( T \) is a continuous linear functional on \( C[0, 1] \), equipped with the sup-norm.

10. True or False: If \( f \) and \( g \) are in \( R[0, 1] \), then
\[
\int_0^1 [f(x) + g(x)]^2 \, dx \leq \int_0^1 [f(x)]^2 \, dx + \int_0^1 [g(x)]^2 \, dx
\]

11. True or False: If \( f \in R[a, b] \), then \( \int_a^b \frac{1}{1+|f(x)|} \, dx \) exists.

12. True or False: If \( f \in R[0, 1] \), then \( \int_0^1 x f(x) \, dx \leq \frac{1}{\sqrt{3}} \left( \int_0^1 |f(x)|^2 \, dx \right)^{\frac{1}{2}} \).
Part II - 52 points. Prove carefully two of the following three theorems. Circle the letters of the two proofs to be counted - no more than two! You may write the proofs below, on the back, or on scratch paper.

A. Prove the Mean Value Theorem for Integrals: Let $f \in C[a, b]$. Prove: There exists $\bar{x} \in [a, b]$ such that $\int_a^b f(x) \, dx = f(\bar{x})(b-a)$. (Hints: Let $m = \min\{f(x) | x \in [a, b]\}$ and $M = \max\{f(x) | x \in [a, b]\}$. You may assume that $C[a, b] \subseteq R[a, b]$.)

B. Suppose $f \in C(a, b)$ and also that $f$ is bounded on $[a, b]$. Prove: $f \in R[a, b]$. (Hint: Use a version of the Darboux Criterion.)

C. Prove that $\| \cdot \|_2$ does satisfy the triangle inequality: $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$. (Hint: Write $\|f + g\|_2^2 = \langle f + g, f + g \rangle$, expand using linearity in each variable, and apply the Cauchy-Schwarz inequality.)