Print Your Name Here: ______________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if it is to be graded.* No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the following 12 questions for 6 points each. **Circle** the numbers of the 8 questions listed below that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: Every bounded sequence in $\mathbb{E}^n$ has a subsequence that is Cauchy.

2. For each of the following sets, state whether it is *open, closed, both,* or *neither.*

a. $A = \left\{\mathbf{x} \in \mathbb{E}^2 \,\middle|\, \|x\| \geq 0\right\}$.

b. $C = \left\{\mathbf{x} \in \mathbb{E}^2 \,\middle|\, x_1 \geq 0 \text{ and } x_2 > 0\right\}$.

c. $E = \left\{\mathbf{x} \in \mathbb{E}^2 \,\middle|\, x_1 + x_2 > 0\right\}$.

3. Find the sets $\overline{\mathbb{Q}^n}$ and $(\mathbb{Q}^n)^\circ$, the closure and the interior of the set $\mathbb{Q}^n$ in $\mathbb{E}^n$.

4. Let $S = \{\mathbf{x} \in \mathbb{E}^2 \mid 0 < \|\mathbf{x}\| \leq 1\}$. True or False: S is compact.

5. Let $T = \{\mathbf{x} \in \mathbb{E}^2 \mid \|\mathbf{x}\| = 1\}$. True or False: S is connected.

6. If $f(\mathbf{x}) = \dfrac{x_1 x_2^2}{x_1^2 + x_2^2}$ either find $\lim\limits_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ or state that this limit does not exist.

7. Is the set $A = \left\{\mathbf{x} \in \mathbb{E}^4 \,\middle|\, x_1 x_4 - x_2 x_3 = 1\right\}$ open, closed, neither, or both?

8. Give an example of a *relatively open* subset A of $D = \{\mathbf{x} \in \mathbb{E}^2 \mid 1 < \|\mathbf{x}\| \leq 2\}$ that is *not open* in $\mathbb{E}^2$.

9. Give an example of a *relatively closed* subset A of $D = \{\mathbf{x} \in \mathbb{E}^2 \mid 1 < \|\mathbf{x}\| \leq 2\}$ that is *not closed* in $\mathbb{E}^2$.

10. True or False: A set $S \subseteq \mathbb{E}^1$ must be connected in $\mathbb{E}^1$ if the set $\mathfrak{S} = \{(x, 0) \in \mathbb{E}^2 \mid x \in S\}$ is connected in $\mathbb{E}^2$.

11. Let $f(x_1) = \begin{cases} \sin \frac{\pi}{x_1} & \text{if } x_1 \in (0, 1], \\ 0 & \text{if } x_1 = 0. \end{cases}$ True or False: The graph G_f is a compact subset of $\mathbb{E}^2$.

12. Let f be the function given in Question #11. True or False: the Graph G_f is connected.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Suppose a vector space V is equipped with an inner product $\langle \cdot, \cdot \rangle$, and suppose we define a corresponding norm by $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. Prove the *Parallelogram Law:* $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

B. Let $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_k \supseteq \ldots$ be a decreasing nest of *nonempty closed* subsets of $\mathbb{E}^n$.

- **(i)** (20) If E_1 is compact, show that $\bigcap_{k=1}^{\infty} E_k \neq \emptyset$. (Hint: Select a point $\mathbf{x}_k \in E_k$ for each $k \in \mathbb{N}$. Cite and use an important theorem about sequences in $\mathbb{E}^n$.)
- **(ii)** (6) Give an example to show it is possible for $\bigcap_{k=1}^{\infty} E_k$ to be empty if E_1 is not compact.

C. Let $f : \mathbb{E}^2 \to \mathbb{R}$ by the formula $f(\mathbf{x}) = \begin{cases} \frac{x_1^2 x_2}{x_1^4 + x_2^2} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$

- **(i)** (6) Show that $f \in \mathcal{C}(\mathbb{E}^2 \setminus \{\mathbf{0}\}, \mathbb{R})$.
- **(ii)** (8) Show that *at the origin* f is continuous as a function of x_1 (with x_2 held fixed at 0) and that f is continuous as a function of x_2 with x_1 held fixed.
- **(iii)** (12) Is $f \in \mathcal{C}(\mathbb{E}^2, \mathbb{R})$? Prove your conclusion.

Solutions and Class Statistics

1. True: The Bolzano-Weierstrass Theorem guarantees the existence of a convergent subsequence.

2. The set A is both open and closed; C is neither open nor closed; E is open.

3. $\overline{\mathbb{Q}^n} = \mathbb{E}^n$ and $(\mathbb{Q}^n)^{\circ} = \emptyset$

4. False, because S is not closed.

5. True: The upper semicircle is connected, being the graph of a continuous function on an interval, as is the lower semicircle. The graphs intersect.

6. The limit is 0.

7. The set A is closed. If $f(x) = x_1x_4 - x_2x_3$ then $f : \mathbb{E}^4 \to \mathbb{R}$ is continuous and $A = f^{-1}\{1\}$, the preimage of a closed set. Also $A \neq \emptyset \neq A^c$ so A cannot be both open and closed, since $\mathbb{E}^4$ is connected.

8. For example, we could let $A = \{\mathbf{x} \in \mathbb{E}^2 \mid \frac{3}{2} < \|\mathbf{x}\| \leq 2\}$.

9. For example, we could let $A = \{\mathbf{x} \in \mathbb{E}^2 \mid 1 < \|\mathbf{x}\| \leq \frac{3}{2}\}$.

10. True. If $S = E \cup F$, a disjoint union of sets neither of which contains any cluster point of the other, then $\mathfrak{S} = \mathfrak{E} \cup \mathfrak{F}$, violating the assumed connectivity of $\mathfrak{S}$. Here $\mathfrak{E} = \{(x, 0) \mid x \in E\}$ and $\mathfrak{F} = \{(x, 0) \mid x \in F\}$.

11. False: The set G_f is not closed.

12. True. If $G_f \subset A \cup B$ with A and B open and disjoint, then the whole graph must lie in the open set that contains the origin. Note that f is continuous on $(0, 1)$.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8				
80-89 (B)	4				
70-79 (C)	2				
60-69 (D)	0				
0-59 (F)	1				
Test Avg	87.9%	%	%	%	%
HW Avg	5.5				
HW Adj	66.5%	%	%	%	%
HW/Test Correl	0.83				