Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Find $\|\mathbf{x} + \mathbf{y}\|$ in terms of $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$.

2. Let $(V, \|\cdot\|)$ be an arbitrary normed vector space with a scalar product on V such that $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$ for all $\mathbf{x} \in V$. If $\|\mathbf{x}\| = 2$ and $\|\mathbf{y}\| = 3$, find $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2$.

3. Express the arbitrary set $S \subseteq \mathbb{E}^n$ as the union of a family of *closed* sets.

4. Let $p \in \mathbb{E}^n$. Express the singleton set $\{p\}$ as the intersection of a *countable* family of *open* sets.

5. Find the *interior*, $(\mathbb{Q}^n)^o$, of $\mathbb{Q}^n$, and the *boundary* $\partial \mathbb{Q}^n$ of $\mathbb{Q}^n$.

6. Let $S = \left\{\mathbf{x}^{(j)} \,\middle|\, j \in \mathbb{N}\right\} \subset \mathbb{E}^n$ be *any* sequence of points in a Euclidean space. Prove or give a counter-example: The set S is a *compact* set if and only if the sequence $\mathbf{x}^{(j)}$ is convergent *to an element of* S.

7. For each of the following subsets of $\mathbb{E}^n$, state whether the set *is* or *is not* compact.

a. $B_r(\mathbf{x})$, with $r > 0$.

b. $\bar{B}_r(\mathbf{x})$, with $r > 0$.

c. $S^{n-1} = \bar{B}_r(\mathbf{x}) \setminus B_r(\mathbf{x})$, with $r > 0$.

8. True or False: The set $E = \left\{\mathbf{x} \,\middle|\, x_1 = 0\right\} \cup \left\{\mathbf{x} \,\middle|\, x_1 x_2 = 1, x_1 > 0\right\}$ is a connected subset of $\mathbb{E}^2$.

9. Let $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_k \supseteq \ldots$ be a decreasing nest of *nonempty compact* subsets of $\mathbb{E}^n$. True or Give a Counterexample: $\bigcap_{k \in \mathbb{N}} E_k \neq \emptyset$.

10. True or False: Every *open* set $S \subseteq \mathbb{E}^n$ can be expressed as the union of a family of mutually *disjoint* open balls.

11. Let $f(x) = \begin{cases} x \cos \frac{\pi}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$ Is the graph $G_f = \{(x, f(x)) \mid 0 \leq x \leq 1\}$ connected?

12. Let $f(x) = \begin{cases} \sin \frac{\pi}{x} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$ Is the graph $G_f = \{(x, f(x)) \mid 0 \le x \le 1\}$ *compact*?

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Show that a sequence of vectors $\mathbf{x}^{(j)} \to \mathbf{x} \in \mathbb{E}^n$ as $j \to \infty$, meaning that $\|\mathbf{x}^{(j)} - \mathbf{x}\| \to 0$ as $j \to \infty$, if and only if the coordinate sequence $x_l^{(j)} \to x_l$ as $j \to \infty$ for each $l = 1, \ldots, n$. (Hint: $\|\mathbf{x}\| = \sqrt{\sum_{l=1}^{n} x_l^2}$, the Euclidean norm in $\mathbb{E}^n$.)

B. Let S be any subset of $\mathbb{E}^n$ and let C be the set of *all cluster points* of S. Prove that $S \cup C$ is a closed set. (Hint: Show that $(S \cup C)^c$, the *complement* of $S \cup C$, is open by letting $p \in (S \cup C)^c$ and showing that there exists $r > 0$ such that $B_r(p) \subseteq (S \cup C)^c$)

C. Let $f(x) = \begin{cases} \sin \frac{\pi}{x} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$ Prove that the graph of f is a connected subset of $\mathbb{E}^2$. (Hint: Try writing $G_f \subseteq A \dot{\cup} B$ with A and B open and disjoint. Prove that G_f must be contained entirely within one of the two open sets.)

Solutions and Class Statistics

1. $\|\mathbf{x}+\mathbf{y}\| = \sqrt{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}$. This is the Pythagorean theorem in any inner product space, even in infinite dimensions.

2. $2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) = 2(4+9) = 26$. This is the *parallelogram law* from Euclidean geometry.

3. $S = \bigcup_{x \in S} \{x\}$

4. $\{p\} = \bigcap_{n \in \mathbb{N}} B_{\frac{1}{n}}(p)$

5. $(\mathbb{Q}^n)^o = \emptyset$ and $\partial \mathbb{Q}^n = \mathbb{E}^n$.

6. Counterexample: $S = \left\{ \mathbf{x}^{(j)} = ((-1)^j, (-1)^j) \,\middle|\, j \in \mathbb{N} \right\} \subset \mathbb{E}^2$ is a finite set, and thus compact. But the sequence $\mathbf{x}^{(j)} = ((-1)^j, (-1)^j)$ does not converge. It is not Cauchy. *This problem has a typo! I should have written True or Give a Counterexample, since proofs are not required in Part I. No one asked during the test so I did not catch this until now.*

7.

a. is *not* compact

b. *is* compact

c. *is* compact

8. False.

9. True.

10. False: that would make every open set disconnected.

11. Yes, G_f is connected because f is continuous on $[0, 1]$.

12. No, G_f is not compact since it is missing cluster points and is not closed.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Here are some general remarks about the proofs I have just read.

A. This is a double implication: It has the form Statement P if and only Statement Q. It is important to state for each of the two proofs you write, which direction you are proving: the "if" or the "only if". The "if" part is the implication from right to left, $P \Leftarrow Q$. It means you are assuming that statement Q is true and proving that statement P follows inescapably from this. The "only if" direction of the proof means that you are assuming that statement P is true and proving that statement Q is true: $P \Rightarrow Q$. By writing down explicitly which side you are assuming and which side you will prove is a consequence, you are making a roadmap for yourself so that you don't get lost or confused along the way. At the same time you are showing me that you understand what you are doing.

B. Remember that every proof on Part II is a graded homework proof, or at least a part of a multipart homework proof. This proof is half of the exercise in which you proved $\bar{S} = S \cup C$. You are asked to prove that $S \cup C$ is closed, thereby showing the half of the exercise that is to establish that $S \cup C \supseteq \bar{S}$. Do not assume the exercise part of which is what you are being asked to prove as a test question. Use the offered hint: it is meant to help you stay on track. Show that there exists $r > 0$ such that $B_r(p) \cap S = \emptyset$. Then show that $B_r(p) \cap C = \emptyset$.

C. The origin is the star-player here. Suppose the origin lies in A, which is open. Thus there is $r > 0$ such that $B_r(\mathbf{0}) \subseteq A$. You need to use the fact that $f|_{(0,\infty)}$ is continuous and so its graph is connected. Use the oscillatory behavior of f near the origin to prove that the restricted graph intersects $B_r(\mathbf{0})$ which lies in A.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	3				
80-89 (B)	2				
70-79 (C)	2				
60-69 (D)	2				
< 60 (F)	0				
Test Avg	80.6%	%	%	%	%
HW Avg	6.7				
HW/Test Correl	0.91	–			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^9$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.