Print Your Name Here: ______________________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Let $f : \mathbb{E}^2 \to \mathbb{R}$ by the formula $f(\mathbf{x}) = \begin{cases} \frac{x_1^2 x_2}{x_1^4 + x_2^2} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$ Does $\lim_{\mathbf{x} \to \infty} f(\mathbf{x})$ exist?

2. Let V be a normed vector space. If $\epsilon > 0$ find the largest $\delta > 0$ such that $\|\mathbf{x} - \mathbf{y}\| < \delta \implies |\|\mathbf{x}\| - \|\mathbf{y}\|| < \epsilon$.

3. Suppose $f : \mathbb{E}^2 \to \mathbb{R}$ has the property that for *each* fixed value $x_2 = b$ the function $f(x_1, b)$ is a continuous function of x_1. Suppose also that for *each* fixed value $x_1 = a$ the function $f(a, x_2)$ is a continuous function of x_2. Does it follow that $f \in \mathcal{C}\left(\mathbb{E}^2, \mathbb{R}\right)$?

4. The continuous function $\phi(t) = (\cos[2\pi t], \sin[2\pi t])$ maps $[0, 1)$ one-to-one and onto the unit circle S_1. True or False: The inverse function ϕ^{-1} is also continuous.

5. Suppose $D \subset \mathbb{E}^n$ is *compact* and $\mathbf{f} \in \mathcal{C}\left(D, \mathbb{E}^m\right)$ is one-to-one. True or False: If $\mathbf{f}(D)$ is connected, then D is connected.

6. Let A and B be in $\mathcal{L}\left(\mathbb{E}^2\right)$ with matrices in the standard basis $[A] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $[B] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Find* $\|A\|, \|B\|$, and $\|A+B\|$.

7. *Give an example* of $T \in \mathcal{L}(\mathbb{E}^2)$ for which $\|T^2\| < \|T\|^2$, and *find* $\|T^2\|$ and $\|T\|^2$.

For Questions 8, 9, and 10 below, let $\mathbf{f} : \mathbb{E}^2 \to \mathbb{E}^2$ ***be defined by*** $\mathbf{f}(\mathbf{x}) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$.

8. Calculate the matrix $[\mathbf{f}'(\mathbf{x})]$ with respect to the standard basis for $\mathbb{E}^2$.

9. Find $\det \mathbf{f}'(\mathbf{x})$.

10. Calculate $D_{\mathbf{v}}\mathbf{f}\left(0, \frac{\pi}{6}\right)$, where $\mathbf{v} = (1, \sqrt{3})$.

11. Suppose $\mathbf{f} : \mathbb{E}^n \to \mathbb{E}^m$ is such that every *directional derivative* $D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ exists for all $\mathbf{v}, \mathbf{x} \in \mathbb{E}^n$. True or False: f must be differentiable on $\mathbb{E}^n$.

12. True or Give a Counterexample: If $f : \mathbb{E}^2 \to \mathbb{E}^1$ is *differentiable* at $\mathbf{0} \in \mathbb{E}^2$ then f must be *continuous* at $\mathbf{0}$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $D \subseteq \mathbb{E}^n$ and $\mathbf{f} : D \to \mathbb{E}^m$. We call $\mathbf{f}$ *uniformly continuous* on D if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{x} - \mathbf{x}'\| < \delta$ implies $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')\| < \epsilon$, for all $\mathbf{x}$ and $\mathbf{x}' \in D$. If D is *compact* and if $\mathbf{f} \in \mathcal{C}(D, \mathbb{E}^m)$, prove that $\mathbf{f}$ is uniformly continuous on D. (Hint: Suppose false and use the Bolzano-Weierstrass Theorem to deduce a contradiction.)

B. Let $L \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$ and $T \in \mathcal{L}(\mathbb{E}^m, \mathbb{E}^k)$, so that $T \circ L \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^k)$.

(**i**) Prove: $\|T \circ L\| \leq \|T\|\|L\|$. (Hint: Recall the definition of the norm a linear transformation.)

(**ii**) Now let $k = m = n$. Denote $T^2 = T \circ T$ and $T^{j+1} = T^j \circ T$ for all $j \in \mathbb{N}$. Show that $\|T^j\| \leq \|T\|^j$.

C. Define $f : \mathbb{E}^2 \to \mathbb{E}^1$ by $f(\mathbf{x}) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } \mathbf{x} \in \mathbb{E}^2 \setminus \{\mathbf{0}\}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$

(**i**) Prove: $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ exist at *each* $\mathbf{x} \in \mathbb{E}^2$. (Suggestion: Each partial derivative is, by definition, an ordinary one-variable derivative, holding the other variable temporarily constant. Using your knowledge of calculus, what can you learn about the two partial derivatives of f if $\mathbf{x} \neq \mathbf{0}$ and then at $\mathbf{x} = \mathbf{0}$?.)

(**ii**) Prove: f is *not* differentiable at $\mathbf{x} = \mathbf{0}$. For 5 bonus points, prove that f *is* differentiable at each $\mathbf{x} \neq \mathbf{0}$.

Solutions and Class Statistics

1. No, the limit does not exist. Along either axis the function is identically 0 but if $\mathbf{x} \to \infty$ along the parabola $x_2 = x_1^2$ then the function is identically $\frac{1}{2}$.

2. $\delta = \epsilon$. Use the inequality $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| \leq \|\mathbf{x} - \mathbf{y}\|$.

3. No, f need not be continuous.

4. False: ϕ^{-1} fails to be continuous at the point $(1, 0$.

5. True, because this function f must have a continuous inverse.

6. $\|A\| = \|B\| = \|A + B\| = 1$.

7. Using the standard basis, let $T(\mathbf{e}_1) = 0,\ T(\mathbf{e}_2) = \mathbf{e}_1$. Then $\|T\| = 1$ and $\|T^2\| = 0$.

8. $[\mathbf{f}'(\mathbf{x})] = \left[\dfrac{\partial f_i}{\partial x_j}\right]_{2\times 2} = \begin{bmatrix} e^{x_1}\cos x_2 & -e^{x_1}\sin x_2 \\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2 \end{bmatrix}$

9. $\det \mathbf{f}'(\mathbf{x}) = e^{2x_1}$

10. $D_{\mathbf{v}}\mathbf{f}\left(0, \dfrac{\pi}{6}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

11. False.

12. True.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Here are some general remarks about the proofs I have just read.

A. Here the key is to *negate* correctly the definition of uniform continuity on D: $\exists \epsilon > 0$ such that no $\delta > 0$ can satisfy the requirement that $x, x' \in D$ with $\|x - x'\| < \delta \implies \|f(x) - f(x')\| < \epsilon$. Let $\delta_n = \frac{1}{n}$ and $\exists x_n, x'_n \in D$ with $\|x_n - x'_n\| < \frac{1}{n}$ yet $\|f(x_n) - f(x'_n)\| \geq \epsilon$. Show that $x_n \to p$ for some $p \in D$ and prove that $x'_n \to p$ as well. Use continuity at p to show that $\|f(x) - f(x')\| \to 0$, which contradicts $\|f(x_n) - f(x'_n)\| \geq \epsilon$.

B. It is important in this problem to make clear how you use the definition of the norm of a linear transformation to justify your steps in the proof.

C. For the first part there are two steps. First use elementary calculus to find *correctly* the two partial derivatives if the point is not the origin. Then one must use the *definition* of the partial derivative to show that both partial derivatives exist at the origin. For the second part one needs to show that f is *not continuous* at the origin. For the bonus credit one must show that the partial derivatives of f are continuous on $\mathbb{E}^2 \setminus \{\mathbf{0}\}$.

Class Statistics

Grade	Test#1	Test#2	Final Exam	Final Grade
90-100 (A)	3	6		
80-89 (B)	2	1		
70-79 (C)	2	0		
60-69 (D)	2	1		
$<$ 60 (F)	0	0		
Test Avg	80.6%	89.1%	%	
HW Avg	6.7	6.2		
HW/Test Correl	0.91	0.94		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^9$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.