1) $y'' - xy' + 4y = 0$; $y(0)=1$, $y'(0)=0$

a) $y = \sum_{k=0}^{\infty} a_k x^k \quad y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

$\underbrace{\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}}_{j=k-2} - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 4 a_k x^k = 0$

$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j$

$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 4 a_k x^k = 0$

$\forall k \geq 0$: $(k+2)(k+1) a_{k+2} = (k-4) a_k$

$$\boxed{a_{k+2} = \frac{k-4}{(k+2)(k+1)} a_k} \quad \forall k \geq 0$$

b) $a_1 = y'(0) = 0 = a_3 = \dots = a_{2k-1} \quad \forall k \geq 1.$

$\boxed{a_0 = y(0) = 1}$

$a_2 = \frac{-4}{2 \cdot 1} = -2$

$a_4 = \frac{-2}{4 \cdot 3}(-2) = \frac{(-1)^2 2^2}{4 \cdot 3} = \frac{4}{4 \cdot 3} = \frac{1}{3}$

$a_6 = 0 = a_8 = \dots = a_{2k} \quad \forall k \geq 3.$

c) $\boxed{y = 1 - 2x^2 + \frac{1}{3} x^4}$

2) (30)

Find gen'l soln to the Cauchy-Euler eq'n

$$x^2y'' + 11xy' + 9y = 0, \quad x>0.$$

(Sol'n $y = \frac{c_1}{x} + \frac{c_2}{x^9}$

Sol'n $y = x^r$: $r(r-1) + 11r + 9 = 0$

$r^2 + 10r + 9 = 0$

$(r+1)(r+9) = 0$

$r = -1, -9$

$$y = \frac{c_1}{x} + \frac{c_2}{x^9}$$

3)

a) $4xy'' + 2y' + y = 0$

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$, $y' = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}$, $y'' = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}$

$$\sum_{k=0}^{\infty} 4(k+r)(k+r-1) a_k x^{k+r-1} + \sum_{k=0}^{\infty} 2(k+r) a_k x^{k+r-1} + \sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

$j = k-1$

$$\sum_{j=-1}^{\infty} 4(j+r+1)(j+r) a_{j+1} x^{j+r} + \sum_{j=-1}^{\infty} 2(j+r+1) a_{j+1} x^{j+r} + \sum_{j=0}^{\infty} a_j x^{j+r} = 0$$

Indicial Eq'n: ($j=-1$) Set Coeff of x^{r-1}: $4(r)(r-1) a_0 + 2r a_0 = 0$, $a_0 \neq 0$

$4r^2 - 2r = 0$

$2r^2 - r = 0$

$(2r-1) r = 0$

$\boxed{r_1 = \frac{1}{2}\ ;\ r_2 = 0}$

b) $\forall j \geq 0$, $4(j+r+1)(j+r) a_{j+1} + 2(j+r+1) a_{j+1} + a_j = 0$

$2(j+r+1)[2(j+r)+1] a_{j+1} = -a_j$

$$\boxed{a_{j+1} = \frac{-a_j}{(2j+2r+2)(2j+2r+1)} \quad (j \geq 0)} \underset{(r_1 = \frac{1}{2})}{=} \frac{-a_j}{(2j+3)(2j+2)}$$

c) Take $a_0 = 1$

$a_1 = \frac{-1}{3 \cdot 2}$

$a_2 = \frac{-a_1}{5 \cdot 4} = \frac{(-1)^2}{5!}$

$\vdots$

$\boxed{a_k = \frac{(-1)^k}{(2k+1)!}}$

$$y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{k+1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\sqrt{x})^{2k+1}$$

$\boxed{y_1 = \sin(\sqrt{x})}$

d) $r = r_2 = 0$

$a_{j+1} = \frac{-a_j}{(2j+2)(2j+1)}$

$a_0 = 1$

$a_1 = \frac{-1}{2 \cdot 1}$

$a_2 = \frac{(-1)^2}{4 \cdot 3} \cdot \frac{1}{2 \cdot 1} = \frac{(-1)^2}{4!}$

$\vdots$

$a_k = \frac{(-1)^k}{(2k)!}$

$$y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (\sqrt{x})^{2k} = \boxed{\cos(\sqrt{x}) = y_2(x)}$$

gen'l soln $\boxed{y = c_1 \sin\sqrt{x} + c_2 \cos\sqrt{x}}$