1. (30) All solutions to Equation (1) below are analytic, and have the form $y = \sum_{k=0}^{\infty} a_k x^k$.

$$y'' - xy' + y = 0 \quad (1)$$

a. (15) Find a recursion formula for all $a_{k+2}$ in terms of $a_k$ and/or $a_{k+1}$.

b. (10) Find all those odd-indexed coefficients $a_{2k-1}$ that must be zero.

c. (5) Write the general solution for $y$ in terms of $a_0$, $a_1$ and all those powers of $x$ up to $x^8$. 
2. (40) The differential equation \(2xy'' + y' + y = 0\) has a regular singular point at \(x = 0\). The Method of Frobenius assures the existence of at least one solution of the form \(y = \sum_{m=0}^{\infty} a_m x^{m+r}\).

a. (15) Find the **indicical equation** and its two roots \(r_1\) and \(r_2\), \(r_1 \geq r_2\), and note that they do not differ by an integer.

b. (15) Find the **recursion relation** that determines \(a_{m+1}\) in terms of \(a_m\) and the general value of \(r\). (Suggestion: If you express \(a_{m+1}\) in terms of \(2a_m\) this will help you to find the factorial in the denominator of the general coefficient \(a_m\) in part (c) below.)

c. (10) **Using the smaller root** \(r_2\), and choosing \(a_0 = 1\) for convenience, find \(a_1, a_2,\) and \(a_3,\) and find the general coefficient \(a_m\) in terms of \(m\). Express the solution \(y_2\) corresponding to \(r_2\) as the sum of a series using the preceding information. (**5 point bonus** if you express \(y_2\) as an elementary function.)
3. (30) Use the steps in (a)-(c) to solve

\[ xy'' - y' - y = 0. \]  \hspace{1cm} (2)

a. (10) Use a new independent variable \( t = 2\sqrt{x} \) to transform Eq. (2) into an equation in \( y, t \) and derivatives of \( y \) with respect to \( t \).

b. (10) Use a new dependent variable \( w = \frac{y}{x} \) to transform the result of (a) into a modified Bessel equation of the form

\[ t^2 \frac{d^2 w}{dt^2} + t \frac{dw}{dt} - (t^2 + \nu^2) w = 0. \]

c. (10) Write the general solution to Equation (2) for \( y \) in terms of \( x \).
Solutions

1. a. \( a_{k+2} = \frac{(k-1)}{(k+2)(k+1)}a_k \) for all \( k \geq 0 \), or any equivalent formulation.

b. \( a_3 = 0 = a_5 = a_7 = \cdots = a_{2k-1} \) for all \( k \geq 2 \).

c. \( y = a_1x + a_0 \left[ 1 - \frac{x^2}{2} - \frac{x^4}{4!} - \frac{3x^6}{6!} - \frac{5 \cdot 3x^8}{8!} - \cdots \right] \). The 3 dots indicate that the series goes on ad infinitum. It is a good thing to write the solution in this form, as the general linear combination of two linearly independent solutions. The solution space for a second order linear homogeneous equation must be two-dimensional. Please do not multiply out factorials!

2. a. The indicial equation is \( r(2r-1) = 0 \) so that \( r_1 = \frac{1}{2} \) and \( r_2 = 0 \).

b. The recursion relation is \( a_{m+1} = \frac{-a_m}{(2m+2r+1)(m+r+1)} = \frac{-2a_m}{(2m+2r+1)(2m+2r+2)} \). The right-most form above is most useful because the denominator leads readily to a factorial expression in the formula for \( a_m \).

c. Using the smaller root \( r_2 = 0 \), and using \( a_0 = 1 \), \( a_1 = -\frac{2}{2!} = -1 \), \( a_2 = \frac{2^2}{3!} \), \( a_3 = -\frac{2^3}{6!} = -\frac{1}{90} \) and \( a_m = (-1)^m \frac{2^m}{(2m)!} \). Thus \( y_2 = \sum_{m=0}^{\infty} \frac{(-1)^m 2^m}{(2m)!} x^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left( \sqrt{2}x \right)^{2m} = \cos \sqrt{2}x \). We remark that the latter expression for \( y_2 \) as an elementary function is not required for full credit.

3. a. \( \frac{d^2y}{dt^2} + \frac{3}{t} \frac{dy}{dt} - y = 0 \). Be careful with the Chain Rule.

b. \( t^2 \frac{d^2w}{dt^2} + t \frac{dw}{dt} - (t^2 + 4)w = 0 \). Note that the complicated formula in the text is useless for a modified Bessel equation. But this shouldn’t matter since I have given you the correct changes of variable here.

c. \( y = x \left[ c_1 I_2(2\sqrt{x}) + c_2 K_2(2\sqrt{x}) \right] \). Of course we need to use the modified Bessel functions here.

Class Statistics

<table>
<thead>
<tr>
<th>% Grade</th>
<th>Test #1</th>
<th>Test #2</th>
<th>Test #3</th>
<th>Final Exam</th>
<th>Final Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>90-100 (A)</td>
<td>4</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
<tr>
<td>80-89 (B)</td>
<td>4</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
<tr>
<td>70-79 (C)</td>
<td>12</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
<tr>
<td>60-69 (D)</td>
<td>3</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
<tr>
<td>0-59 (F)</td>
<td>2</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
<tr>
<td>Test Avg</td>
<td>76.7%</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
<tr>
<td>HW Avg</td>
<td>97.6%</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
</tbody>
</table>