

Print Your Name Here: _____

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side,* and write your name at the top of the scratch sheet if you will turn it in for grading.
- **Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** problems: maximum total score = 100.

1. (30) All solutions to Equation (1) below, with the given initial conditions, are analytic, and have

the form $y = \sum_{k=0}^{\infty} a_k x^k$.

$$y'' - xy' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 0 \quad (1)$$

a. (15) Find a *recursion formula* for all a_{k+2} in terms of a_k and/or a_{k+1} .

b. (5) Find all the *odd-indexed* coefficients a_{2k-1} . Be sure to use the initial conditions!

c. (10) Find all the *even-indexed* coefficients a_{2k} , and write out the complete solution to the given initial value problem.

2. (20) Find the general solution to the Cauchy-Euler Equation

$$x^2y'' + 11xy' + 9y = 0.$$

3. (50) The differential equation $4xy'' + 2y' + y = 0$ has a regular singular point at $x = 0$. The Method of Frobenius assures the existence of at least one solution of the form $y = \sum_{k=0}^{\infty} a_k x^{k+r}$.

- a. (15) Find the *indicial equation* and its roots r_1 and r_2 , $r_1 > r_2$.

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- b. (15) Find the *recursion relation* that determines a_{k+1} in terms of a_k using the *general value* of r .
- c. (10) Using the **larger** root r_1 , and choosing $a_0 = 1$ for convenience, find the *general coefficient* a_k in terms of k . Express the solution y_1 corresponding to r_1 as the sum of a series using the preceding information. If possible, please express y_1 an elementary function of x .
- d. (10) Now use the smaller root r_2 and choose $a_0 = 1$. Find a_k in terms of k , and find a second solution y_2 . Write the *general solution*, expressing it *if possible* as an elementary function.

Solutions

Please remember to SHOW ALL WORK: ANSWERS ALONE ARE NOT SUFFICIENT. If your work does not fit into the spaces provided, continue on the scratch sheets and be sure to number the work on the scratch sheets with the problem number and problem part. And state "see work continued on scratch sheet" on the printed pages. I was not strict about this on this first test, but I will be later.

1.

a. (15) $a_{k+2} = \frac{(k-4)a_k}{(k+2)(k+1)}$ for all $k \geq 0$, or any equivalent formulation. One must be careful with indexing and with algebra.

b. (5) $a_1 = 0 = a_3 = a_5 = a_7 = \dots = a_{2k-1}$ for all $k \geq 1$. One *must* use the initial conditions! Every engineer **must** know what the first two coefficients in a Maclaurin series are in terms of the function and its derivative at $x = 0$.

c. (10) $y = 1 - 2x^2 + \frac{1}{3}x^4$. Thus the solution to the given initial value problem is a polynomial. Note that the solution is unique since this is an initial value problem.

2. (20) $y = \frac{c_1}{x} + \frac{c_2}{x^9}$. Some students forgot all about the special, easy form of the solution to a Cauchy-Euler equation.

3.

a. (15) The *indicial equation* is $2r^2 - r = 0$ so that $r_1 = \frac{1}{2}$ and $r_2 = 0$. One must be very careful re-indexing so that x appears to the same power in each series. Be careful with the starting point for the new index.

b. (15) The *recursion relation* is $a_{k+1} = -\frac{a_k}{(2k+2+2r)(2k+1+2r)}$, $k \geq 0$.

c. (10) Using the *larger* root $r_1 = \frac{1}{2}$, and using $a_0 = 1$, $a_k = \frac{(-1)^k}{(2k+1)!}$, using r_1 for r in part (b). Thus

$$y_1 = \sum_{k=0}^{\infty} (-1)^k \frac{(\sqrt{x})^{2k+1}}{(2k+1)!} = \sin \sqrt{x}, \text{ which is an elementary function of } x.$$

d. (10) $y_2 = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{(\sqrt{x})^{2k}}{(2k)!} = \cos \sqrt{x}$. The general solution is $y = c_1 y_1 + c_2 y_2 = c_1 \sin \sqrt{x} + c_2 \cos \sqrt{x}$, which is an elementary function.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8				
80-89 (B)	2				
70-79 (C)	0				
60-69 (D)	4				
0-59 (F)	3				
Test Avg	79.4%	%	%	%	%
HW Avg	95.76%	%	%		
HW/Test Correl.	0.22				

The Correlation Coefficient is the cosine of the angle between two data vectors in \mathbb{R}^{17} —one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a weak positive correlation with performance on the homework. This suggests that 7 students may be using shortcuts to get high homework scores, instead of maintaining a notebook with detailed hand written solutions to each assigned problem. How a student does the Webassign homework in this course makes a big difference. Please follow my instructions as to how to solve the homework problems.