1 \[ \|P_n\|: \text{The Norm of the } n\text{th Legendre Polynomial} \]

We will prove that \( \|P_n\| = \sqrt{\frac{2}{2n + 1}} \) for each nonnegative integer \( n \), leaving numerous details for the reader to check.

We will use Rodriguez’s Formula\(^1\), established in class: \( P_n(x) = \frac{1}{2^n n!} \frac{d^n u}{dx^n} \), where \( u = (x^2 - 1)^n \).

We will apply integration by parts repeatedly.

\[
(2n)!^2 \int_{-1}^{1} P_n^2 dx = \int_{-1}^{1} u^{(n)} u^{(n)} dx = u^{(n)} u^{(n-1)} \bigg|_{-1}^{1} - \int_{-1}^{1} u^{(n+1)} u^{(n-1)} dx = \int_{-1}^{1} u^{(n+1)} u^{(n-1)} dx
\]

\[
= \ldots = (-1)^n (2n)! \int_{-1}^{1} (x^2 - 1)^n dx = (2n)! \int_{-1}^{1} (1 - x^2)^n dx
\]

\[
= (A) (2n)! \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1} \theta d\theta \overset{(B)}{=} (2n)! \frac{(2^n n!)^2}{(2n + 1)!}
\]

where step (A) is a trigonometric substitution, and step (B) is explained as follows.

We prove by induction that \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1} \theta d\theta = 2 \int_{0}^{\frac{\pi}{2}} \cos^{2n+1} \theta d\theta = \frac{(2^n n!)^2}{(2n + 1)!} \). The case \( n = 0 \) is easy and is left as an exercise. Now assume that the formula is true for \( n - 1 \) and use integration by parts to prove the validity of the formula for \( n \).

Finally, the steps above establish the claimed norm for \( P_n \).

2 \text{ Zeros of the Bessel Functions: Heuristic Explanation} \]

Consider the Bessel equation for any \( \nu \geq 0 \):

\[ x^2 y'' + xy' + (x^2 - \nu^2) y = 0. \]

Make the substitution \( y = \frac{u}{\sqrt{x}} \), where \( u \) will be the new dependent variable, replacing \( y \), and \( x \) remains the independent variable. Show that this transforms the general Bessel equation into

\[ u'' + \left(1 + \frac{1 + \nu^2}{x^2}\right) u = 0. \]

Observe that if \( x \gg 0 \), which is read as \( x \) is very much bigger than zero, then this equation is quite similar to \( u'' + u = 0 \). The latter equation is solved by \( \cos x \) and by \( \sin x \), which are functions that oscillate endlessly between positive and negative values. In fact, the equation tells us that \( u \) accelerates toward negative values whenever \( u \) is positive, and vice versa. This suggests strongly that \( u \), and also \( y \), must cross the \( x \)-axis infinitely often. Details of a rigorous argument can be found in Watson’s book.

3 \text{ Norm of the Bessel Function } J_n, \ n \geq 0\]

Note that \( n \) does not need to be an integer in this section. We will prove that

\[ \|J_n(\lambda_{mn} x)\|^2 = \frac{R^2}{2} J_{n+1}^2(\lambda_{mn} R). \]

\(^1\)These and other topics can be studied further in Watson’s classic treatise, \textit{Bessel Functions}.
Here $\lambda_{mn}R = a_{mn}$, the $m$th zero of $J_n$, and $||J_n(\lambda_{mn}x)||^2 = \int_0^R J_n^2(\lambda_{mn}x)\,dx$.

Denote $\phi_n(x) = J_n(\lambda x)$ for general $\lambda \in \mathbb{R}$. It follows that

$$[x\phi_n'(x)]' + \left(\frac{-n^2}{x} + \lambda^2 x \right) \phi_n(x) = 0.$$  

Now multiply by $2x\phi_n'(x)$:

$$\left([x\phi_n'(x)]^2\right)' + (\lambda^2 x^2 - n^2) [\phi_n(x)]' = 0.$$  

Next integrate both sides:

$$[x\phi_n'(x)] \bigg|_0^R = -\int_0^R (\lambda^2 x^2 - n^2) [\phi_n(x)]' \,dx.$$

Recall the identity

$$[x^{-n}J_n(x)]' = -x^{-n}J_{n+1}(x)$$ and proceed as follows.

$$-nx^{-n-1}J_n(x) + x^{-n}J_n'(x) = -x^{-n}J_{n+1}(x)$$

$$-nJ_n(x) + xJ_n'(x) = -xJ_{n+1}(x).$$  

Next replace $x$ by $\lambda x$:

$$\lambda x \frac{\phi_n'(x)}{\lambda} = n\phi_n(x) - \lambda x\phi_{n+1}(x).$$

Combining the latter calculations we find from the left side of Equation (*) that

$$[x\phi_n'(x)]^2 \bigg|_0^R = [n\phi_n(x) - \lambda x\phi_{n+1}(x)]^2 \bigg|_0^R$$

$$(\text{Now set } \lambda = \lambda_{mn})$$

$$= \lambda_{mn}^2 R^2 J_{n+1}^2(\lambda_{mn}R) \text{ since } J_n(\lambda_{mn}R) = 0 \text{ if } n \geq 1.$$  

Integrate (*) on the right by parts:

$$= -\left(\lambda_{mn}^2 x^2 - n^2\right) J_n^2(\lambda_{mn}x) \bigg|_0^R + 2\lambda_{mn}^2 \int_0^R xJ_n^2(\lambda_{mn}x) \,dx.$$

It follows that $\int_0^R J_n^2(\lambda_{mn}x)\,dx = \frac{R^2}{2} J_{n+1}^2(\lambda_{mn}R)$, which concludes the proof.