# 1 $\|P_n\|$: The Norm of the $n$th Legendre Polynomial

We will prove that $\|P_n\| = \sqrt{\dfrac{2}{2n+1}}$ for each nonnegative integer $n$, leaving numerous details for the reader to check.

We will use Rodriguez's Formula[1], established in class: $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n u}{dx^n}$, where $u = \left(x^2-1\right)^n$. We will apply integration by parts repeatedly.

$$(2^n n!)^2 \int_{-1}^{1} P_n^2\,dx = \int_{-1}^{1} u^{(n)}u^{(n)}\,dx = u^{(n)}u^{(n-1)}\Big|_{-1}^{1} - \int_{-1}^{1} u^{(n+1)}u^{(n-1)}\,dx = \int_{-1}^{1} u^{(n+1)}u^{(n-1)}\,dx$$

$$= \ldots = (-1)^n(2n)!\int_{-1}^{1}\left(x^2-1\right)^n\,dx = (2n)!\int_{-1}^{1}\left(1-x^2\right)^n\,dx$$

$$\overset{(A)}{=} (2n)!\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1}\theta\,d\theta \overset{(B)}{=} (2n)!\frac{2\left(2^n n!\right)^2}{(2n+1)!}$$

where step (A) is a trigonometric substitution, and step (B) is explained as follows.

We prove by induction that $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1}\theta\,d\theta = 2\int_{0}^{\frac{\pi}{2}} \cos^{2n+1}\theta\,d\theta = \frac{\left(2^n n!\right)^2}{(2n+1)!}$. The case $n=0$ is easy and is left as an exercise. Now assume that the formula is true for $n-1$ and use integration by parts to prove the validity of the formula for $n$.

Finally, the steps above establish the claimed norm for $P_n$.

# 2 Zeros of the Bessel Functions: Heuristic Explanation

Consider the Bessel equation for any $\nu \ge 0$:

$$x^2y'' + xy' + \left(x^2-\nu^2\right)y = 0.$$

Make the substitution $y = \dfrac{u}{\sqrt{x}}$, where $u$ will be the new dependent variable, replacing $y$, and $x$ remains the independent variable. Show that this transforms the general Bessel equation into

$$u'' + \left(1 + \frac{\frac{1}{4}-\nu^2}{x^2}\right)u = 0.$$

Observe that if $x >> 0$, which is read as $x$ is *very much bigger* than zero, then this equation is quite similar to $u'' + u = 0$. The latter equation is solved by $\cos x$ and by $\sin x$, which are functions that oscillate endlessly between positive and negative values. In fact, the equation tells us that $u$ *accelerates* toward negative values whenever $u$ is positive, and vice versa. This suggests strongly that $u$, and also $y$, must cross the $x$-axis infinitely often. Details of a rigorous argument can be found in Watson's book.

# 3 Norm of the Bessel Function $J_n$, $n \ge 0$

Note that $n$ does not need to be an integer in this section. We will prove that

$$\|J_n(\lambda_{mn}x\|^2 = \frac{R^2}{2}J_{n+1}^2(\lambda_{mn}R).$$

[1]These and other topics can be studied further in Watson's classic treatise, *Bessel Functions*.

Here $\lambda_{mn}R = a_{mn}$, the $m$th zero of $J_n$, and $\|J_n(\lambda_{mn}x\|^2 = \int_0^R J_n^2(\lambda_{mn}x)\,x\,dx$.

Denote $\phi_n(x) = J_n(\lambda x)$ for general $\lambda \in \mathbb{R}$. It follows that

$$[x\phi_n'(x)]' + \left(\frac{-n^2}{x} + \lambda^2 x\right)\phi_n(x) = 0. \text{ Now multiply by } 2x\phi_n'(x):$$

$$\left([x\phi_n'(x)]^2\right)' + \left(\lambda^2x^2 - n^2\right)\left[\phi_n^2(x)\right]' = 0. \text{ Next integrate both sides:}$$

$$[x\phi_n'(x)]\Big|_0^R = -\int_0^R \left(\lambda^2x^2 - n^2\right)\left[\phi_n^2(x)\right]'\,dx. \quad (*)$$

Recall the identity

$$\left[x^{-n}J_n(x)\right]' = -x^{-n}J_{n+1}(x) \text{ and proceed as follows.}$$

$$-nx^{-n-1}J_n(x) + x^{-n}J_n'(x) = -x^{-n}J_{n+1}(x)$$

$$-nJ_n(x) + xJ_n'(x) = -xJ_{n+1}(x). \text{ Next replace } x \text{ by } \lambda x:$$

$$\lambda x\frac{\phi_n'(x)}{\lambda} = n\phi_n(x) - \lambda x\phi_{n+1}(x).$$

Combining the latter calculations we find from the left side of Equation (*) that

$$\begin{aligned}[x\phi_n'(x)]^2\Big|_0^R &= [n\phi_n(x) - \lambda x\phi_{n+1}(x)]^2\Big|_0^R \text{ (Now set } \lambda = \lambda_{mn}) \\ &= \lambda_{mn}^2R^2J_{n+1}^2(\lambda_{mn}R) \text{ since } J_n(\lambda_{mn}R) = 0 \text{ if } n \geq 1. \text{ Integrate (*) on the right by parts:} \\ &= \underbrace{-\left(\lambda_{mn}^2x^2 - n^2\right)J_n^2(\lambda_{mn}x)\Big|_0^R}_{=0} + 2\lambda_{mn}^2\int_0^R xJ_n^2(\lambda_{mn}x)\,dx.\end{aligned}$$

It follows that $\int_0^R J_n^2(\lambda_{mn}x)\,x\,dx = \frac{R^2}{2}J_{n+1}^2(\lambda_{mn}R)$, which concludes the proof.