1. (30) The function \( f(x) = 1 - x \) on the domain \((0, \pi)\) has a Fourier half-range expansion

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin nx.
\]

(a) (5) Evaluate \( \int_0^\pi \sin^2 nx \, dx \), which is the square of the norm of the function \( \sin nx \).

(b) (20) Find the value of \( b_n \) for all positive integers \( n \).

c. (5) The sum of the series in Equation (1) is equal for all \( x \) in \((-\pi, 0)\) to a linear function \( f_o(x) = ax + b \). Find \( a \) and \( b \).
2. (40) Consider the following Sturm-Liouville problem on the interval [0, 1]

\[ y'' - \lambda y = 0; \quad y'(0) = 0, \quad y(1) = 0. \]  \hspace{1cm} (2)

a. (5) Showing all work, determine all eigenfunctions for (2) corresponding to \( \lambda = 0 \).

b. (10) Showing all work, determine all positive eigenvalues \( \lambda = \mu^2 \) and the corresponding eigenfunctions for (2).

c. (25) Showing all work, determine all negative eigenvalues \( \lambda = -\mu^2 \) and the corresponding eigenfunctions for (2).
3. (30) Let \( f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \\ -1, & -1 \leq x < 0 \end{cases} \). Thus \( f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \) where the \( n \)th Legendre polynomial

\( P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \) and \( \|P_n\|^2 = \frac{2}{2n + 1} \).

a. (6) Find the numerical value of \( c_0 \).

b. (6) Find the numerical value of \( c_1 \).

c. (16) Express the value of \( c_n \) for all odd \( n \) in terms of the value of the derivative of some order of \( (x^2 - 1)^n \) at \( x = 0 \).

d. (2) Find the numerical value of \( c_n \) if \( n \) is even. Explain briefly.
1.

a. $\int_{0}^{\pi} \sin^2 nx \, dx = \frac{\pi}{2}$.

b. We use integration by parts to calculate $b_n = \frac{2}{\pi} \int_{0}^{\pi} (1 - x) \sin nx \, dx = \frac{2}{n} [1 + (\pi - 1)(-1)^n]$. 

c. Since the sum of a sine series is an odd function $f_n(x) = -f(-x) = -(1 - x) = -1 - x$ for all $x$ in $(-\pi, 0)$). That is, $a = -1 = b$. See Fig. (1).

\[ \text{Figure 1: Fourier Series (1) to N=15} \]

2.

a. If $\lambda = 0$ then $y = ax + b$ and the boundary conditions force $y \equiv 0$. Thus there are no eigenfunctions and $\lambda = 0$ is not an eigenvalue.

b. There are no eigenvalues $\lambda = \mu^2 > 0$ or eigenfunctions, since if $\lambda = \mu^2$ then $y = ae^{\mu x} + be^{-\mu x}$. One should write out the boundary conditions and show that these force $a = b = 0$ so that $y \equiv 0$ and thus there is no eigenfunction.

c. $\lambda_n = -\left(n + \frac{1}{2}\right)^2 \pi^2$, $y_n = \cos \left(n + \frac{1}{2}\right) \pi x$, $n = 0, 1, 2, \ldots$.

3. The partial sum of this Legendre series appears in Fig. (2)

a. $c_0 = 0$.

b. $c_1 = \frac{3}{2}$.

c. If $n$ is odd the integrand becomes even so the integral is twice its value from 0 to 1. Thus $c_n = \frac{-2n + 1}{2n+1} \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \bigg|_{x=0}$.

d. If $n$ is even $P_n$ is even and $f$ is an odd function, making the integrand odd over the interval $[-1, 1]$, so that $c_n = 0$.

Class Statistics

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Figure 2: Legendre Series to $N=25$, $f(x)=\text{Sign}(x)$