Print Your Name Here: ____________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** problems: maximum total score = 100.

1. (30) Let $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$. Find the Fourier coefficients a_0, $a_n,\ n \ge 1$, and $b_n,\ n \ge 1$ in the expansion $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

2. (40) Consider the regular Sturm-Liouville problem

$$\frac{d}{dx}\left[(1+x^2)y'\right] + \frac{\lambda}{1+x^2}y = 0,\ \ y(0) = 0 = y(1). \tag{1}$$

a. (10) Introduce a new independent variable $\theta = \tan^{-1} x,\ 0 \le \theta \le \frac{\pi}{4}$. Use the chain rule to rewrite Eq.(1) in terms of y and its derivatives with respect to θ. (This should produce a familiar equation.)

b. (20) *Solve* the equation found in part (a) for y in terms of θ and use the boundary conditions for the interval $0 \le \theta \le \frac{\pi}{4}$ to *find all* the eigenvalues λ_n. *Find* the corresponding eigenfunctions y_n and express y_n in terms of x.

c. (10) Find the weight function $p(x)$ and write out the integral that expresses the orthogonality relation $\langle y_n(x), y_m(x)\rangle_p = 0$ if $m \neq n$. Do not work out the integration.

3. (30) The set $\{P_n(x) \mid n = 0, 1, 2, \ldots\}$ of Legendre polynomials $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ is a complete orthogonal system on the interval $[-1, 1]$ with $\|P_n\|^2 = \frac{2}{2n+1}$. Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & -1 \le x \le 0 \end{cases}$.

Expand $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n P_n(x)$ by finding the following information.

a. (10) Find c_0, c_1, c_2, c_3.

b. (15) Use integration by parts to find a general formula for c_n if $n \ge 2$ in terms of the evaluation of a derivative of some order of $(x^2-1)^n$ at a certain point.

c. (5) What can you say about the coefficients c_n if n is *odd* and $n > 2$? Please explain.

Solutions

1. $a_0 = \dfrac{\pi}{4}$, $a_n = \dfrac{(-1)^n - 1}{\pi n^2}$, $b_n = \dfrac{(-1)^{n+1}}{n}$ if $n \geq 1$. One may note that $a_n = 0$ if $n > 1$ is even. Also, a_0, being the constant term of the Fourier series, must be the average value of the function on the given interval $[-\pi, \pi]$.

2. This was problem 12.5/#4 in the homework.

a. $\dfrac{d^2y}{d\theta^2} + \lambda y = 0$, $y(\theta) = 0$ at $\theta = 0$ and at $\theta = \frac{\pi}{4}$. Notice that you were given the proper range for θ corresponding to the given endpoints for x.

b. $\lambda_n = 16n^2$, $n = 1, 2, 3, \ldots$, and $y_n = \sin(4n \tan^{-1} x)$.

c. $p(x) = \frac{1}{1+x^2}$ and $\int_0^1 \sin(4n \tan^{-1} x) \sin(4m \tan^{-1} x) \frac{1}{1+x^2}\, dx = 0$ if $m \neq n$. Notice that if you make the change of variables to $\theta = \tan^{-1} x$ then the constant 4 makes sense in terms of the given range for θ.

3. This is very similar to 12.6/#4 in the homework, and to the Legendre series examples we worked out during class time and during the review class.

a. $c_0 = \dfrac{1}{4}$, $c_1 = \dfrac{1}{2}$, $c_2 = \dfrac{5}{16}$, $c_3 = 0$.

b. $c_n = \dfrac{2n+1}{2^{n+1} n!} \dfrac{d^{n-2}}{dx^{n-2}} (x^2 - 1)^n \Big|_{x=0}$ for $n \geq 2$.

c. (5) The *odd-indexed* coefficients $c_n = 0$ if $n > 2$. This is because the *odd function* $g(x) = \dfrac{2n+1}{2^{n+1} n!} \dfrac{d^{n-2}}{dx^{n-2}} (x^2 - 1)^n \Big|_{x=0}$ must be zero at $x = 0$ since $g(0) = -g(-0) = -g(0)$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	2			
80-89 (B)	1	2			
70-79 (C)	1	2			
60-69 (D)	0	1			
0-59 (F)	0	2			
Test Avg	91.6%	74.8%	%	%	%
HW Avg	96.73%	96.9%	%		
Test/HW Correl		0.75			