1. (30) Let \( f(x) = 1 - |x| \) on \([-1, 1]\). Find all the Fourier coefficients in the Fourier series
\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x).
\]
2. (30) Consider the square wave \( f(x) = \begin{cases} 
1 & \text{if } 0 \leq x < \pi \\
0 & \text{if } -\pi \leq x < 0 
\end{cases} \) with period \( 2\pi \). Express \( f(x) \) as the sum of a Fourier series.
3. (40) Consider the (Cauchy-Euler) differential equation

\[ x^2 y'' + xy' + \lambda y = 0, \ \lambda > 0. \]  (1)

a. (15) Find the general real-valued solution to (1). (Hint: Recall that \( x^r = e^{r \ln x} \) and use Euler’s formula.)

b. (15) Find all the eigenvalues \( \lambda_n \) and corresponding eigenfunctions \( y_n \) satisfying the boundary conditions \( y'(1) = 0 \) and \( y'(e) = 0 \) on the derivative of \( y \).

c. (10) Put the equation (1) into self-adjoint form and identify the weight-function \( p(x) \) for which

\[ \int y_m(x)y_n(x)p(x)dx = 0 \]  whenever \( m \neq n \). Now write out the latter integral using your solution functions and your \( p(x) \) and use elementary calculus to reduce this integral to one that is already known to be zero.
Solutions

1. \( a_0 = \text{avg}_{[-1,1]}(f) = \frac{1}{2} \) and \( b_n = 0 \) since \( f \) is even. If \( n \geq 1 \) we have \( a_n = \int_{-1}^{1} (1 - |x|) \cos n\pi x \, dx = \frac{2}{n\pi} \left( \frac{1}{n\pi} - (-1)^n \right) \) if \( n \) even and \( \frac{4}{n^2\pi} \) if \( n \) odd. Here we have used the evenness of \( f \) to simplify \( |x| \) by doubling the integral from 0 to 1. Remember that \( |x| = -x \) if \( x < 0 \). One needs to use integration by parts. Many students introduced unnecessary work leading to errors because of failing to use the benefits of an even or an odd integrand.

2. Since \( f(x) - \frac{1}{2} \) is an odd function it has a pure sine series: \( f(x) - \frac{1}{2} = \sum_{n=1}^{\infty} b_n \sin nx \), where

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} \frac{1}{2} \sin nx \, dx.
\]

Thus \( f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} (-1)^n \sin nx \).

3. This is almost the same as Webassign 12.5 problem 2.

   a. (15) Substituting \( y = x^r \) into (1), we find \( r = \pm i\sqrt{\lambda} \) and \( x^{\sqrt{\lambda}} = e^{i\sqrt{\lambda}\ln x} = \cos(\sqrt{\lambda}\ln x) + i\sin(\sqrt{\lambda}\ln x) \). This yields the general real solution \( y = c_1 \cos(\sqrt{\lambda}\ln x) + c_2 \sin(\sqrt{\lambda}\ln x) \).

   b. (15) \( \lambda_n = (n\pi)^2 \) and \( y_n = \cos(n\pi \ln x) \), \( n = 1, 2, 3, \ldots \).

c. (10) The self-adjoint form is \( (xy')' + \frac{\lambda}{x} y = 0 \) and \( p(x) = \frac{1}{x} \). Finally, \( \int_{1}^{e} \cos(n\pi u) \cos(n\pi u) \, du = 0 \) where we have substituted \( u = \ln x \).

Class Statistics

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The Correlation Coefficient is the cosine of the angle between two data vectors in 18-dimensional space: one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.