Print Your Name Here: ______________________________

- **INSTRUCTIONS:**
- **Show all work:** ***Answers without work are not sufficient.*** We can give credit *only* for what you write! ***Indicate clearly if you continue on the back side***, and write your **name at the top of the scratch sheet** if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** required problems: maximum total score = 100, or 110 with the *optional Bonus Problem.*

1. (25) Solve the differential equation $xy''+5y'+xy = 0$ as follows. Introduce a new *dependent* variable $w = x^2y$ to convert the equation to a Bessel equation of the form $x^2w'' + xw' + (x^2 - \nu^2)w = 0$ for a suitable value of ν.

2. (25) To solve the differential equation

$$y'' + \left(e^{2x} - 4\right) y = 0 \tag{1}$$

introduce a *new independent* variable $t = e^x$.

a. (10) Write both y' and y'' entirely in terms of y, t, and the derivatives of y with respect to t. (Remember: y'' in (1) denotes the second derivative with respect to x, not with respect to t.)

b. (10) Rewrite equation (1) entirely in terms of y, t, and the derivatives of y with respect to t.

c. (5) Solve the equation obtained in part (b) for y in terms of t and then express y as a function of x.

3. (50) Let $f(x) = |x|$ on $(-\pi, \pi)$. The Fourier series for $f(x) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$.

a. (10) Find the value of α_0.

b. (10) Find the value of β_n for all $n = 1, 2, 3, \ldots$.

c. (20) Find the value of α_n for all $n = 1, 2, 3, \ldots$.

d. (10) The Fourier series you have found converges to a periodic function on the whole real line. Sketch the graph of the function to which the Fourier series converges on the interval $(-3\pi, 3\pi)$.

e. OPTIONAL 10 POINT BONUS: Find the sum $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$. (Hint: use Parseval's identity, or just remember that the Fourier series is a series of mutually orthogonal functions.)

Solutions

Please remember to SHOW ALL WORK: ANSWERS ALONE ARE NOT SUFFICIENT. If your work does not fit into the spaces provided, continue on the scratch sheets and be sure to number the work on the scratch sheets with the problem number and problem part. And state "see work continued on scratch sheet" on the printed pages.

1. $y = \dfrac{c_1 J_2(x) + c_2 Y_2(x)}{x^2}$. Note that since 2 is an integer, $J_{-2} = (-1)^2 J_2$, so that the set $\{J_2, J_{-2}\}$ is not a linearly independent set. That is why Y_2, a Bessel function of the second kind, is needed. Some students had difficulty substituting for y and its derivatives in terms of w and its derivatives. Note that $y = x^{-2}w$, $y' = x^{-2}w' - 2x^{-3}w$, and $y'' = x^{-2}w'' - 4x^{-1}w' + 6x^{-4}w$. Substituting into the original equation and simplifying leads to the correct value $\nu = 2$. Then express w in terms of x and then y in terms of x, which is the *required solution* of the given differential equation. Some students did not actually write the solution, which is essential.

2.

a. $y' = \dfrac{dy}{dt}\dfrac{dt}{dx} = t\dfrac{dy}{dt}$ and $y'' = t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt}$

b. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + (t^2 - 4)y = 0.$

c. $y = c_1 J_2(t) + c_2 Y_2(t) = c_1 J_2(e^x) + c_2 Y_2(e^x)$. It is important to write the general solution, which requires again the use of a Bessel function of the second kind, since 2 is an integer.

3.

a. (10) $\alpha_0 = \frac{\pi}{2}$, which is the average value of $f(x)$ on $(-\pi, \pi)$. It is *very important* to understand that $|x| \neq x$ on $(-\pi, \pi)$. One function is even and the other is odd! The integral is the sum of the areas of two right triangles, or π^2, and then divide by the length, 2π, of the interval to get the average value of f on the interval.

b. (10) $\beta_n = 0$ for all $n = 1, 2, 3, \ldots$ since f is an even function. There is no need to evaluate an integral here!

c. (20) $\alpha_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |x| \cos nx\, dx = \dfrac{2}{\pi}\int_0^{\pi} x \cos nx\, dx = \dfrac{2}{\pi n^2}[(-1)^n - 1]$ for all $n = 1, 2, 3, \ldots$. It will save you much work if you recognize that the function $|x| \cos nx$ in the first integrand is an *even* function.

d. (10)

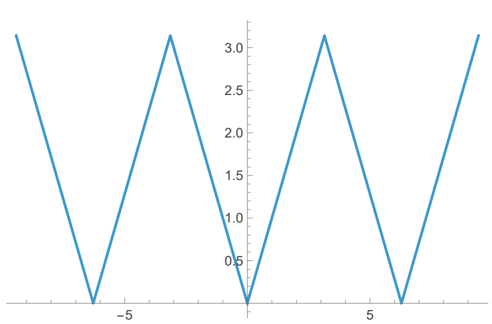

Figure 1: Periodic Sum of the Fourier Series.

e. OPTIONAL 10 POINT BONUS: $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}$. Congratulations to the student who got this!

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	4			
80-89 (B)	2	6			
70-79 (C)	0	4			
60-69 (D)	4	1			
0-59 (F)	3	1			
Test Avg	79.4%	81.6 %	%	%	%
HW Avg	95.76%	94.61%	%		
HW/Test Correl.	0.22	0.67			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{17}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strong positive correlation with performance on the homework.