Print Your Name Here: ____________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** problems: maximum total score = 100.

1. (30) USe the steps below to find all solutions of the special form $u(x,t) = F(x)G(t)$ to the heat equation $u_t = u_{xx}$, $0 \le x \le 1$, with the *boundary conditions* $u_x(0,t) = 0 = u_x(1,t)$, $t \ge 0$, representing *insulated ends* of a rod of length 1.

a. (10) Use the separation constant $-\lambda$ to *find* two ordinary differential equations for the functions $F(x)$ and $G(t)$, *stating* the boundary conditions on F.

b. (10) Find *all* eigenvalues λ_n and *all* corresponding eigenfunctions $F_n(x)$ for the resulting Sturm-Liouville problem for $F(x)$.

c. (10) Find all corresponding functions $G_n(t)$ and write all product solutions of the form $u_n(x,t) = F_n(x)G_n(t)$.

2. (30) Consider the Laplace equation $u_{xx} + u_{yy} = 0$ on the *semi-infinite* region $0 \le x \le 1$ and $0 \le y < \infty$. The boundary conditions are $u(0, y) = 0 = u(1, y)$ for all $y \ge 0$, $u(x, 0) = x,\ 0 < x < 1$, and $u(x, y)$ *must remain bounded* as $y \to \infty$.

a. (10) For product solutions $u(x, y) = F(x)G(y)$, use the separation constant $-\lambda$ to write ordinary differential equations for $F(x)$ and $G(y)$, including boundary conditions on F and the condition on $G(y)$ as $y \to \infty$.

b. (10) Find all the eigenvalues λ_n, the corresponding eigenfunctions $F_n(x)$, and the corresponding functions $G_n(y)$, remembering the condition on $G(y)$.

c. (10) *Express* the solution $u(x, y) = \sum_{n=0}^{\infty} b_n F_n(x) G_n(y)$, *finding the numerical value* of b_n needed to satisfy the boundary condition $u(x, 0) = x$, $0 < x < 1$.

3. (40) Consider the 2-dimensional wave equation $u_{tt} = u_{xx} + u_{yy}$ on the rectangular domain $0 \le x \le a,\ 0 \le y \le b$. Use the *boundary conditions* $u(x, 0, t) = 0 = u(x, b, t) = u(0, y, t) = u(a, y, t)$ for all $t \ge 0$, $0 \le x \le a,\ 0 \le y \le b$, and the *initial conditions* $u(x, y, 0) = f(x, y),$ and $u_t(x, y, 0) = 0$. Let $u(x, y, t) = F(x)G(y)H(t)$ be a product solution.

a. (10) Use the separation constant $-\lambda$ to write an ordinary differential equation for $F(x)$ with *boundary* conditions and find the eigenvalues λ_n and the corresponding eigenfunctions $F_n(x)$.

b. (10) Use the separation constant $-\kappa$ to write an ordinary differential equation for $G(y)$ with *boundary* conditions and find the eigenvalues κ_m and the corresponding eigenfunctions $G_m(y)$.

c. (10) Use λ_n and κ_m to *write and solve* an ordinary differential equation for $H_{m,n}(t)$ with *initial* condition on $H'_{m,n}(t)$ and *write* the general product solution $u_{m,n}(x, y, t)$.

d. (10) Now satisfy the initial condition on $u(x, y, t) = \sum_{m,n=1}^{\infty} B_{m,n} u_{m,n}(x, y, t)$ by expressing $B_{m,n}$ as a double integral involving the initial displacement function $f(x, y)$.

Solutions

1.

a. $F''(x) + \lambda F(x) = 0$, with $F'(0) = 0 = F'(1)$ and $G'(t) + \lambda G(t) = 0$.

b. $\lambda_n = n^2\pi^2$ and $F_n(x) = \cos n\pi x$, $n = 0, 1, 2, 3, \ldots$. Note that $n = 0$ is necessary in this problem.

c. $G_n(t) = e^{-n^2\pi^2 t}$ and $u_n(x,t) = e^{-n^2\pi^2 t}\cos n\pi x$, $n = 0, 1, 2, \ldots$.

2.

a. $F''(x) + \lambda F(x) = 0$, with $F(0) = 0 = F(1)$ and $G''(y) - \lambda G(y) = 0$ where $G(y)$ is *bounded* as $y \to \infty$. A common *error* was to write $G(0) = x$, which is *impossible*, since G is a function of y alone. Remember that the variables have been *separated*.

b. $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \ldots$, $F_n(x) = \sin n\pi x$, and $G_n(y) = e^{-n\pi y}$.

c. $u(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\pi y}\sin n\pi x$, where $b_n = \dfrac{(-1)^{n+1}2}{n\pi}$.

3.

a. $\lambda_n = \left(\dfrac{n\pi}{a}\right)^2$, $F_n(x) = \sin\left(\dfrac{n\pi}{a}\right)x$, $n = 1, 2, 3, \ldots$.

b. $\kappa_m = \left(\dfrac{m\pi}{b}\right)^2$, $G_m(y) = \sin\left(\dfrac{m\pi}{b}\right)y$, $m = 1, 2, 3, \ldots$.

c. $u_{m,n}(x,y,t) = \sin\left(\dfrac{n\pi x}{a}\right)\sin\left(\dfrac{m\pi y}{b}\right)\cos\left(\pi t\sqrt{\left(\dfrac{n}{a}\right)^2 + \left(\dfrac{m}{b}\right)^2}\right)$.

d. $B_{m,n} = \dfrac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\dfrac{n\pi x}{a}\sin\dfrac{m\pi y}{b}\,dydx$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	2	3		
80-89 (B)	1	2	3		
70-79 (C)	1	2	1		
60-69 (D)	0	1	1		
0-59 (F)	0	2	0		
Test Avg	91.6%	74.8%	85.9%	%	%
HW Avg	96.73%	96.9%	98.1%		
Test/HW Correl		0.75	0.61		