Print Your Name Here: ____________________

- **INSTRUCTIONS:**
- **Show all work:** ***Answers without work are not sufficient.*** We can give credit *only* for what you write! ***Indicate clearly if you continue on the back side***, and write your **name at the top of the scratch sheet** if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **three (3)** required problems and an optional 5 point bonus question 3e: maximum total score = 100, or 105 with the bonus question.

1. (20) The differential equation $y'' - 2xy' + 2ny = 0$, for n an arbitrary non-negative integer, has polynomial solutions $H_n(x)$ on the whole real line. You are *not* asked to find these polynomials, $H_n(x)$! Transform this equation into its self-adjoint form $(r(x)y')' + 2np(x)y = 0$, thereby finding the positive weight function $p(x)$. If $n \neq m$, write the integral $\int_{-\infty}^{\infty} H_n(x)H_m(x)\,p(x)dx = 0$, filling in the correct weight function $p(x)$.

2. (20) Use separation of variables, with the *separation constant* $-\lambda$, to find *all* solutions to the partial differential equation $xu_x = yu_y$ that split into the form $u(x, y) = F(x)G(y)$.

3. (60) A thin metal rod lies on the interval $[0, L]$ of the x-axis. *Use the steps shown below* to solve the heat equation

$$u_t = \kappa^2 u_{xx}$$

with the *boundary conditions* $u_x(0, t) = 0$ and $u_x(L, t) = 0$, representing *insulated ends*, and the *initial condition* $u(x, 0) = f(x)$.

a. (10) We seek *split* solutions of the form $u(x, t) = F(x)G(t)$ to the heat equation. Separate variables, using the *separation constant* $-\lambda$, to write differential equations for F and for G corresponding to an eigenvalue λ .

b. (10) Find the eigenfunction $u_0(x, t) = F_0(x)G_0(t)$ corresponding to $\lambda = 0$.

c. (25) Find all the negative eigenvalues $\lambda = -\mu^2 < 0$ and the corresponding eigenfunctions $u_n(x, t) = F_n(x)G_n(t)$. *(Continued on next page)*

d. (15) Write the final solution in the form $u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t)$, using the eigenfunctions you have already found. Use your knowledge of Fourier series to express the terms $u_0(x,t)$ and the coefficients a_n in terms of integrals, so as to satisfy the initial condition $u(x,0) = f(x)$.

e. (**5 point Bonus!**) Find $\lim_{t\to\infty} u(x,t)$ and state how this relates to the insulated endpoints and the initial temperature distribution $f(x)$.

Solutions

Please remember to SHOW ALL WORK: ANSWERS ALONE ARE NOT SUFFICIENT. If your work does not fit into the spaces provided, continue on the scratch sheets and be sure to number the work on the scratch sheets with the problem number and problem part. And state "see work continued on scratch sheet" on the printed pages.

1. The self-adjoint form of the given equation is $\left(e^{-x^2}y'\right)' + 2n\left(e^{-x^2}\right)y = 0$, so that $p(x) = e^{-x^2}$ is the desired weight function. That is $\int_{-\infty}^{\infty} H_n(x)H_m(x)\,e^{-x^2}dx = 0$if $m \neq n$. Many students did not remember what we learned about integrating factors, which are very common in the study of differential equations. Multiplication by the integrating factor must yield an equation equivalent to the given equation. See the WebAssign homework for secgion 12.5, question number 5, for this specific problem.

2. $u(x,y) = C(xy)^{-\lambda}$ with arbitrary constant C, for each real number λ. Note that every value for λ yields a solution. There are no boundary conditions here! This was WebAssign problem 3 for section 13.1. The equation was a very simple first order Cauchy-Euler equation with solutions of the form $F(x) = x^r$ and a similar expression for $G(y)$. Many of the students who got the right answer worked too hard–not recognizing the Cauchy-Euler equation.

3.

a. (10) $F''(x) + \lambda F(x) = 0$ with the boundary conditions $F'(0) = 0 = F'(L)$, and $G'(t) + \lambda\kappa^2 G(t) = 0$. Many students botched the insulated ends as the boundary condition.

b. (10) $u_0(x,t) = a_0$, a constant function, corresponding to $\lambda = 0$.

c. (25) $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ corresponding to $u_n(x,t) = e^{-\left(\frac{n\pi\kappa}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$.

d. (15) $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi\kappa}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$, where $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$, the average value of f on the interval $[0,L]$, and the coefficients $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, for all positive integers n.

e. (5 point Bonus!) $\lim_{t\to\infty} u(x,t) = a_0 = \frac{1}{L}\int_0^L f(x)\,dx$, the average value of the initial temperature distribution $f(x)$ This makes sense since the bar is insulated and there is no other source of heat except from the initial temperature distribution.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	4	6		
80-89 (B)	2	6	2		
70-79 (C)	0	4	5		
60-69 (D)	4	1	2		
0-59 (F)	3	1	0		
Test Avg	79.4%	81.6 %	83.6%	%	%
HW Avg	95.76%	94.61%	91.9 %		
HW/Test Correl.	0.22	0.67	0.68		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{17}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strong positive correlation with performance on the homework.